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Bifurcation from stability to instability for a free boundary problem arising in a tumor model, I

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1. Introduction

Mathematical models of tumor growth, which consider the tumor tissue as a density of proliferating cells, have been developed and studied in many papers; see [2,3,5–9,14,17–20] and the references given there. Most of the models discuss the case of radially symmetric tumors. Since tumors grown in vitro have a nearly spherical shape, it is important to determine whether radially symmetric tumors are asymptotically stable.

Let $\Omega(t)$ denote the tumor domain at time t, and p the pressure within the tumor resulting from the proliferation of the tumor cells. The density of the cells, c, depends on the concentration of nutrients, σ, and, assuming that this dependence is linear, we simply identify c with σ. We also assume a linear dependence of the proliferation rate S on σ:

$$S = \mu(\sigma - \widetilde{\sigma}) \qquad (\widetilde{\sigma} > 0)$$

where $\widetilde{\sigma}$ is a threshold concentration and μ is a parameter expressing the "intensity" of the expansion by mitosis (if $\sigma > \widetilde{\sigma}$) or shrinkage by apoptosis (if $\sigma < \widetilde{\sigma}$) within the tumor. The function σ satisfies the diffusion equation:

$$\sigma_t - \Delta\sigma + \sigma = 0 \quad \text{in } \Omega(t). \tag{1.1}$$

The pressure p is related to the velocity $\boldsymbol{V}$ of the concentration σ, and, assuming Darcy's law in the tissue, we have $\boldsymbol{V} = -\nabla p$. Since, by conservation of mass, $\mathrm{div}\boldsymbol{V} = S$, we obtain for the pressure p the equation

$$\Delta p = -\mu(\sigma - \widetilde{\sigma}) \quad \text{in } \Omega(t). \tag{1.2}$$

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As in the papers cited above, σ and p satisfy the boundary conditions:

$$\sigma = 1 \quad \text{on } \partial\Omega(t) \qquad (1 > \widetilde{\sigma}), \tag{1.3}$$

$$p = \kappa \quad \text{on } \partial\Omega(t) \tag{1.4}$$

where κ is the mean curvature ($\kappa > 0$ if $\Omega(t)$ is a ball). Furthermore,

$$V_n = -\frac{\partial p}{\partial n} \quad \text{on } \partial\Omega(t) \tag{1.5}$$

where n is the outward normal and V_n is the velocity of the free boundary $\partial\Omega(t)$ in the direction n.

We finally prescribe initial conditions:

$$\sigma\Big|_{t=0} = \sigma_0 \quad \text{in } \Omega(0), \text{ where } \Omega(0) \text{ is given.} \tag{1.6}$$

It was proved in [14] (under the preceding assumption that $0 < \widetilde{\sigma} < 1$) that there exists a unique radially symmetric stationary solution to (1.1)–(1.5), and it is given by

$$\sigma_S(r) = \frac{R_S}{\sinh R_S}\frac{\sinh r}{r}, \quad p_S(r) = C - \mu\sigma_S(r) + \frac{\mu}{6}\widetilde{\sigma}r^2 \tag{1.7}$$

where $C = 1/R + \mu - \mu\widetilde{\sigma}R^2/6$ and R_S is uniquely determined by the equation

$$\frac{1}{R_S^2}(R_S \coth R_S - 1) = \frac{\widetilde{\sigma}}{3}. \tag{1.8}$$

In [5] it was proved that if μ is sufficiently small, say $\mu < \overline{\mu}$, then the stationary solution is *asymptotically stable* with respect to any small perturbation. By this we mean that if the initial conditions are of the form

$$\begin{aligned} &\partial\Omega(0) : r = R_S + \varepsilon\rho_0(\theta, \varphi), \\ &\sigma\Big|_{t=0} = \sigma_S(r) + \varepsilon w_0(r, \theta, \varphi) \end{aligned} \tag{1.9}$$

where ρ_0 and w_0 are bounded functions, then, for any ε with $|\varepsilon|$ sufficiently small, there exists a unique solution of (1.1)–(1.6) for all $t > 0$, and, as $t \to \infty$,

$$\partial\Omega(t) \to \text{ sphere, say } |x - a| = R_S \tag{1.10}$$

where $a = O(\varepsilon)$ (In fact, the convergence was proved to be exponentially fast). The goal of the present work is to determine the maximal interval $0 < \mu < \mu_*$ for which this asymptotic stability is valid.

It was proved in [12] that there exists a sequence of symmetry-breaking branches of stationary solutions of (1.1)–(1.5) bifurcating from an increasing sequence $\mu = \mu_n \equiv \mu_n(R_S)$ $(n = 2, 3, 4, \cdots)$ with free boundary

$$r = R_S + \varepsilon Y_{n,0}(\theta) + O(\varepsilon^2)$$

where $Y_{n,0}$ is the spherical harmonic of order $(n,0)$. (In the 2-d case this was proved earlier in [15]). The μ_n are expressed in terms of the Bessel functions I_m,

$$\begin{aligned}\mu_n &= \mu_n(R_S)\\ &= \frac{n[n(n+1)-2]I_{1/2}(R_S)}{2R_S^3 I_{3/2}(R_S)[I_{5/2}(R_S)/I_{3/2}(R_S) - I_{n+3/2}(R_S)/I_{n+1/2}(R_S)]},\end{aligned} \tag{1.11}$$

and $\mu_n(R_S) < \mu_{n+1}(R_S)$. It follows that the stability result (1.10) cannot hold for $\mu = \mu_2(R_S)$.

In order to tackle the asymptotic stability we shall first consider the linear stability of the radially symmetric stationary solution. This means the following: we substitute

$$\begin{aligned}&\sigma(r,\theta,\varphi,t) = \sigma_S(r) + \varepsilon w(r,\theta,\varphi,t)\\ &p(r,\theta,\varphi,t) = p_S(r) + \varepsilon q(r,\theta,\varphi,t)\\ &\partial\Omega(t):\ r = R_S + \varepsilon\rho(\theta,\varphi,t)\end{aligned}$$

into (1.1)–(1.5) and collect the ϵ-order terms. We obtain a linear system called the *linearization* of (1.1)–(1.5) about the stationary solution. It consists of two linear partial differential equations for w, q and one ordinary differential equation for ρ, with the initial conditions as in (1.9), namely,

$$\begin{aligned}&\rho(\theta,\varphi,0) = \rho_0(\theta,\varphi),\\ &w(r,\theta,\varphi,0) = w_0(r,\theta,\varphi).\end{aligned}$$

Because the original problem is invariant under translation of the origin, if we want to keep the origin fixed, we need to impose some constraints on the initial perturbations. For the linearized problem these constraints are automatically satisfied if

$$\begin{aligned}&\rho_0(\theta,\varphi) \textit{ and } w_0(r,\theta,\varphi) \textit{ do not contain sphereical har-}\\ &\textit{monics of order } (1,m),\ m=-1,0,1.\end{aligned} \tag{1.12}$$

For the purpose of dealing with the nonlinear stability, it is advantageous not to make any a priori restrictions on ρ_0, w_0. But then we can only speak of stability in the sense that

$$\rho(\theta,\varphi,t) - \sum_{m=-1}^{1} a_m Y_{1,m} \to 0 \qquad t\to\infty, \tag{1.13}$$

where the a_m are constants; the a_m vanish if (1.12) is satisfied.

The stationary solution is said to be *linearly stable* if the solution of the linearized problem satisfies (1.13) as $t\to\infty$; the stationary solution is said to be *linearly unstable* if for at least one set of initial data $\rho_0(\theta,\varphi)$, $w_0(r,\theta,\varphi)$,

$$\operatorname{diam}(\partial\Omega(t)) \to \infty \text{ for a sequence } t = t_n \uparrow \infty.$$

In this paper we prove that

there exists a function $\mu_* = \mu_*(R_S)$ such that the stationary solution is linearly stable if $\mu < \mu_*(R_S)$, and linearly unstable if $\mu > \mu_*(R_S)$. (1.14)

One might expect, as in other bifurcation problems for free boundary problems [13], $\mu_*(R_S)$ to coincide with the first bifurcation point $\mu_2(R_S)$. Interestingly enough, this is *not* always the case. In fact we shall prove that

there exists a positive number R^* such that $\mu_*(R_S) < \mu_2(R_S)$ if $R_S < R^*$ and $\mu_*(R_S) = \mu_2(R_S)$ if $R_S > R^*$; (1.15)

R^* is approximately 0.1588.

What happens here is that, for $R_S > R^*$, at the bifurcation point $\mu_*(R_S) = \mu_2(R_S)$ one "eigenvalue" which determines the stability of the free boundary problem crosses the imaginary axis at the origin, whereas if $R_S < R^*$, a pair of "eigenvalues" cross the imaginary axis at $\mu_*(R_S)$. We shall explain this in more details in Remark 6.1.

Building upon the estimates derived in the present paper, we shall establish in a subsequent paper the asymptotic stability of the stationary solution for all $\mu \in (0, \mu_*(R_S))$.

The structure of the paper is as follows. In Section 3 we introduce the linearization of the system about the stationary solution $(R_S, \sigma_S(r))$, and its Laplace transform. It will then become clear that the linear stability and instability when $\rho_0 = \overline{\rho} Y_{n,m}(\theta, \rho)$ (where $Y_{n,m}$ is the spherical harmonic of order (n, m)) should depend the location of the zeros of a certain function $h_n(s) = h_n(s, \mu, R_S)$ defined in Section 3. The study of the zeros of $h_n(s, \mu, R_S)$ as a function of the parameter μ is carried out in Section 4. We show that, for $n \neq 1$, the zeros of $h_n(s, \mu, R_S)$ lie in Re $s < 0$ if $\mu < \mu_n^*(R_S)$ and at least one zero lie in Re $s > 0$ if $\mu > \mu_n^*(R_S)$; the case $n = 1$ is treated as an exceptional case. In Section 5 we establish the stability result (1.14) with $\mu_*(R_S) = \min(\mu_0^*(R_S), \mu_2^*(R_S))$ and by the estimates derived in Section 4 it follows that (1.14) holds. In Section 6 we compare $\mu_0^*(R_S)$ with $\mu_2^*(R_S)$. Finally, in Section 2 we have collected various results on the Bessel functions which are needed in the subsequent sections.

Remark 1.1 The results of this paper extend to the more general system

$$c\sigma_t = \Delta\sigma + \lambda\sigma,$$
$$\Delta p = -\mu(\sigma - \widetilde{\sigma})$$

with boundary conditions

$$\sigma = \overline{\sigma}, \quad p = \gamma\kappa, \quad V_n = -\frac{\partial p}{\partial n}.$$

Indeed, by change of variables

$$x \to x/\sqrt{\lambda}, \quad t \to t\gamma/\lambda,$$

one can reduce this system to the system (1.1)–(1.5) with σ_t replaced by $\widetilde{c}\sigma_t$ in (1.1), and with

$$\mu, \overline{\sigma}, \widetilde{\sigma}, c \text{ replaced, respectively, by } \mu\lambda\overline{\sigma}/\gamma, 1, \widetilde{\sigma}/\overline{\sigma} \text{ and } \widetilde{c} = c/\gamma\sqrt{\lambda}.$$

Since replacing σ_t by $\widetilde{c}\sigma_t$ in (1.1) causes only trivial changes in our analysis, we have taken, for simplicity, $\widetilde{c} = 1$.

In the sequel, when there is no confusion, we shall denote the stationary solution $(R_S, \sigma_S(r))$ briefly by $(R, \sigma(r))$.

2. Bessel functions

In the sequel we shall need several known identities and some new inequalities for the modified Bessel function $I_m(\xi)$, where ξ is a complex variable and m any nonnegative real number. For the convenience we collect all these facts in this section.

$$I_m(\xi) = \Big(\frac{\xi}{2}\Big)^m \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(m+k+1)} \Big(\frac{\xi}{2}\Big)^{2k}, \tag{2.1}$$

$$I_{1/2}(\xi) = \sqrt{\frac{2}{\pi\xi}}\ \sinh\xi, \tag{2.2}$$

$$I_{3/2}(\xi) = \sqrt{\frac{2}{\pi\xi}}\ \Big\{ -\frac{\sinh\xi}{\xi} + \cosh\xi \Big\}, \tag{2.3}$$

$$I_{5/2}(\xi) = \sqrt{\frac{2}{\pi\xi}}\ \Big\{ \Big[\frac{3}{\xi^2} + 1\Big] \sinh\xi - \frac{3}{\xi}\cosh\xi \Big\}, \tag{2.4}$$

$$I_m''(\xi) + \frac{1}{\xi} I_m'(\xi) - \Big(1 + \frac{m^2}{\xi^2}\Big) I_m(\xi) = 0, \tag{2.5}$$

$$I_m'(\xi) + \frac{m}{\xi} I_m(\xi) = I_{m-1}(\xi),\ m \geq 1, \tag{2.6}$$

$$I_m'(\xi) - \frac{m}{\xi} I_m(\xi) = I_{m+1}(\xi),\ m \geq 0, \tag{2.7}$$

$$I_{m-1}(\xi) - I_{m+1}(\xi) = \frac{2m}{\xi} I_m(\xi),\ m \geq 1, \tag{2.8}$$

$$I_m(\xi) = \sqrt{\frac{2}{\pi\xi}}\ e^{\xi}\Big[1 - \frac{4m^2-1}{8\xi} + O(|\xi|^{-2})\Big], \tag{2.9}$$

$$|\arg(\xi)| < \frac{\pi}{2} - \delta,\ |\xi| \to +\infty.$$

Throughout this paper we shall be working with the functions

$$P_n(\xi) = \frac{I_{n+3/2}(\xi)}{\xi I_{n+1/2}(\xi)}, \qquad n = 0, 1, 2, 3, \cdots. \tag{2.10}$$

From the preceding properties of the Bessel functions we immediately derive:

$$P_0(\xi) = \frac{1}{\xi}\coth\xi - \frac{1}{\xi^2}, \tag{2.11}$$

$$P_1(\xi) = \frac{1}{\xi\coth\xi - 1} - \frac{3}{\xi^2}, \tag{2.12}$$

$$P_n(\xi) = \frac{1}{2n+3} - \frac{\xi^2}{(2n+3)^2(2n+5)} + O(|\xi|^4),\ |\xi| \to 0, \tag{2.13}$$

$$P_n(\xi) = \frac{1}{\xi} - \frac{n+1}{\xi^2} + O(|\xi|^{-3}),\ |\arg(\xi)| < \frac{\pi}{2} - \delta,\ |\xi| \to +\infty. \tag{2.14}$$

We shall also need the very useful formula which holds for all nonnegative real numbers n ([11, 7.9 on Page 61]):

$$\frac{J_{n+1}(z)}{J_n(z)} = \sum_{m=1}^{\infty} \frac{-2z}{z^2 - j_{n,m}^2}, \qquad I_m(iz) = i^m J_m(z), \tag{2.15}$$

where $j_{n,m}$ are the mth real positive zeros of $J_n(x)$. These zeros satisfy the inequalities ([11, 7.9 on Page 59], or [1, 9.5.2 on page 370])

$$j_{n,1} < j_{n+1,1} < j_{n,2} < j_{n+1,2} < j_{n,3} < j_{n+1,3} < \cdots, \tag{2.16}$$

the first positive zero lies in the interval ([11, 7.9 on Page 60])

$$\sqrt{n(n+2)} < j_{n,1} < \sqrt{2(n+1)(n+3)}, \tag{2.17}$$

and ([21])

$$j_{n,m} - j_{n,m-1} > \pi \quad \text{if } n > \frac{1}{2}, \quad j_{n,m} - j_{n,m-1} < \pi \quad \text{if } n < \frac{1}{2}. \tag{2.18}$$

Written in terms of the P_n function, the equation (2.15) becomes

$$P_n(\xi) = 2\sum_{m=1}^{\infty} \frac{1}{\xi^2 + j_{n+1/2,m}^2}. \tag{2.19}$$

Lemma 2.1 *For any $n \geq 0$,*

$$P_n(\xi) = \frac{1}{\xi^2 P_{n+1}(\xi) + (2n+3)}, \tag{2.20}$$

$$P_n(0) = \frac{1}{2n+3}, \tag{2.21}$$

$$\frac{d}{d\xi}P_n(\xi) = \frac{1}{\xi} - \frac{2n+3}{\xi}P_n(\xi) - \xi P_n^2(\xi), \tag{2.22}$$

$$|P_n(\xi)| \leq \sqrt{2}P_n(|\xi|) \leq \frac{\sqrt{2}}{2n+3} \quad \textit{for } |\arg(\xi)| \leq \pi/4. \tag{2.23}$$

Proof. By (2.8),

$$P_n(\xi) = \frac{I_{n+3/2}(\xi)}{\xi[I_{n+5/2}(\xi) + \frac{2n+3}{\xi} I_{n+3/2}(\xi)]} = \frac{1}{\xi^2 P_{n+1}(\xi) + (2n+3)},$$

which proves (2.20). The assertion (2.21) follows immediately from (2.20). Differentiating $\xi P_n(\xi) = \dfrac{I_{n+3/2}(\xi)}{I_{n+1/2}(\xi)}$ and using (2.6), (2.7), we obtain

$$\begin{aligned} P_n(\xi) + \xi \frac{d}{d\xi} P_n(\xi) &= \frac{I'_{n+3/2}(\xi) I_{n+1/2}(\xi) - I_{n+3/2}(\xi) I'_{n+1/2}(\xi)}{I^2_{n+1/2}(\xi)} \\ &= \frac{1}{I^2_{n+1/2}(\xi)} \Big[\Big(I_{n+1/2}(\xi) - \frac{n+3/2}{\xi} I_{n+3/2}(\xi) \Big) I_{n+1/2}(\xi) \\ &\qquad - I_{n+3/2}(\xi) \Big(I_{n+3/2}(\xi) + \frac{n+1/2}{\xi} I_{n+1/2}(\xi) \Big) \Big] \\ &= 1 - (2n+2) P_n(\xi) - \xi^2 P_n^2(\xi). \end{aligned}$$

This implies (2.22).

Finally, if $\xi = re^{i\theta}$ and $|\theta| \le \pi/4$, then $\cos 2\theta \ge 0$, and by (2.19)

$$\begin{aligned} |P_n(\xi)| &\le 2 \sum_{m=1}^{\infty} \frac{1}{|r^2(\cos 2\theta + i \sin 2\theta) + j^2_{n+1/2,m}|} \\ &\le 2 \sum_{m=1}^{\infty} \frac{1}{\sqrt{r^4 + 2r^2 \cos 2\theta \; j^2_{n+1/2,m} + j^4_{n+1/2,m}}} \\ &\le 2 \sum_{m=1}^{\infty} \frac{\sqrt{2}}{r^2 + j^2_{n+1/2,m}} \;=\; \sqrt{2} P_n(|\xi|) < \frac{\sqrt{2}}{2n+3}. \qquad \square \end{aligned}$$

Lemma 2.2

$$\frac{d}{dr}\{r P_1(r)\} > 0, \; r > 0. \tag{2.24}$$

Proof.

$$\begin{aligned} \frac{d}{dr}\{r P_1(r)\} &= \frac{r^2 - \sinh^2 r}{(r \cosh r - \sinh r)^2} + \frac{3}{r^2} \\ &= \frac{r^4 - r^2 \sinh^2 r + 3r^2 \cosh^2 r + 3 \sinh^2 r - 6r \sinh r \cosh r}{r^2 (r \cosh r - \sinh r)^2}. \end{aligned}$$

Denote the numerator by $c(r)$. Then $c(0) = c'(0) = 0$. Setting $c_1(r) = c'(r)/r$, we compute $c_1(0) = c_1'(0) = c_1''(0) = c_1'''(0) = 0$, and $c_1^{(4)}(r) = 32r \sinh(2r) > 0$. Hence $c(r) > 0$ and (2.24) follows. $\square$

Lemma 2.3 *For r real and positive and $n \geq 0$,*

$$P_n(r) > P_{n+1}(r),\ n \geq 0, \tag{2.25}$$

$$\frac{d}{dr}P_n(r) < 0, \tag{2.26}$$

$$\begin{aligned}\frac{2n+5}{r^2+(2n+3)(2n+5)} &< P_n(r)\\ &< \frac{r^2+(2n+5)(2n+7)}{2(2n+5)r^2+(2n+3)(2n+5)(2n+7)},\end{aligned} \tag{2.27}$$

$$\frac{1}{r} - \frac{n+1}{r^2} < P_n(r) < \frac{1}{r} - \frac{n+1}{r^2} + \frac{n^2+n+1}{2r^3}. \tag{2.28}$$

Proof. If $\xi = r$ is real and positive, then (2.19) and (2.16) imply (2.25). From (2.19) and (2.16) we also immediately deduce the inequality (2.26).

Since $P_{n+1}(r) < P_{n+1}(0) = 1/(2n+5)$, using (2.20) we get

$$\begin{aligned}P_n(r) = \frac{1}{r^2P_{n+1}(r)+(2n+3)} &> \frac{1}{r^2/(2n+5)+(2n+3)}\\ &= \frac{2n+5}{r^2+(2n+3)(2n+5)}.\end{aligned}$$

Using this lower bound for P_{n+1}, we see that

$$P_n(r) = \frac{1}{r^2P_{n+1}(r)+(2n+3)} < \frac{1}{r^2\Big(\dfrac{2n+7}{r^2+(2n+5)(2n+7)}\Big)+(2n+3)},$$

which establishes (2.27).

To prove (2.28), recall, by (2.22), that

$$\frac{d}{dr}P_n(r) + \frac{2n+3}{r}P_n(r) + rP_n^2(r) - \frac{1}{r} = 0.$$

If we compare P_n with the function $\overline{P} \equiv \frac{1}{r} - \frac{n+1}{r^2}$ which satisfies

$$\begin{aligned}&\frac{d}{dr}\overline{P}(r) + \frac{2n+3}{r}\overline{P}(r) + r\overline{P}^2(r) - \frac{1}{r} = -\frac{n(n+1)}{r^3} < 0,\\ &\overline{P}(n+1) = 0 < P_n(n+1),\end{aligned}$$

we immediately deduce $P_n(r) > \overline{P}(r)$ for $r > n+1$. We clearly also have $P_n(r) > 0 > \overline{P}(r)$ for $0 < r < n+1$ so that the first inequality of (2.28) follows. Similarly, a comparison with the function $\widetilde{P} \equiv \frac{1}{r} - \frac{n+1}{r^2} + \frac{n^2+n+1}{2r^3}$, which satisfies

$$\begin{aligned}&\frac{d}{dr}\widetilde{P}(r) + \frac{2n+3}{r}\widetilde{P}(r) + r\widetilde{P}^2(r) - \frac{1}{r} = \frac{1}{r^3}\Big(\frac{n^2+n+1}{2r} - 1\Big)^2 > 0,\\ &\widetilde{P}(r) > P_n(r) \quad \text{for } 0 < r < 1,\end{aligned}$$

yields the second inequality in (2.28). □

3. The linearized problem

Set

$$\begin{aligned}&\sigma(r,\theta,\varphi,t)=\sigma_S(r)+\varepsilon w(r,\theta,\varphi,t)\\&p(r,\theta,\varphi,t)=p_S(r)+\varepsilon q(r,\theta,\varphi,t)\\&\partial\Omega(t):\ r=R_S+\varepsilon\rho(\theta,\varphi,t)\end{aligned}\tag{3.1}$$

and, for brevity, denote R_S by R, then (cf. [16])

$$\begin{aligned}&V_n=\varepsilon\rho_t+O(\varepsilon^2),\\&\kappa=\frac{1}{R}-\frac{\varepsilon}{R^2}\Big(\rho+\frac{1}{2}\Delta_\omega\rho\Big)+O(\varepsilon^2),\end{aligned}$$

where

$$\Delta_\omega\rho=\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial\rho}{\partial\theta}\Big)+\frac{1}{\sin^2\theta}\frac{\partial^2\rho}{\partial\varphi^2}.$$

Substituting these quantities into (1.1)–(1.5) and collecting only the linear terms in ε, we obtain the linearized system (cf. [5])

$$w_t-\Delta w+w=0\quad\text{ in } B_R\times\{t>0\},\tag{3.2}$$

$$w(R,\theta,\varphi,t)=-\lambda\rho(\theta,\varphi,t)\quad\text{for } t>0\qquad(\lambda=\frac{\partial}{\partial r}\sigma_S(r)\Big|_{r=R}),\tag{3.3}$$

$$\Delta q=-\mu w\quad\text{ in } B_R\times\{t>0\},\tag{3.4}$$

$$q(R,\theta,\varphi,t)=-\frac{1}{R^2}\Big(\rho+\frac{1}{2}\Delta_\omega\rho\Big),\tag{3.5}$$

$$\frac{d\rho}{dt}=-\frac{\partial^2 p_S}{\partial r^2}\Big|_{r=R}\rho-\frac{\partial q}{\partial r}\Big|_{r=R}\quad\text{for } t>0,$$

where $B_R=\{r<R\}$. Using (1.7) and (1.8) we find that

$$\frac{\partial^2 p_S}{\partial r^2}\Big|_{r=R}=-\mu(1-\widetilde{\sigma}),$$

Thus the last equation can also be written as

$$\frac{d\rho}{dt}=\mu(1-\widetilde{\sigma})\rho-\frac{\partial q}{\partial r}\Big|_{r=R}\quad\text{for } t>0.\tag{3.6}$$

From (1.9) we also have the initial conditions

$$w\Big|_{t=0}=w_0(r,\theta,\varphi)\quad\text{for } 0\le r\le R,\quad\rho\Big|_{t=0}=\rho_0(\theta,\varphi).\tag{3.7}$$

Theorem 3.1 *If $(w_0,\rho_0)\in C^{2+2\alpha/3}(\overline{B}_R)\times C^{4+\alpha}(\partial B_R)$, then there exists a unique solution of (3.2)–(3.7) for all $t>0$.*

Proof. For $\overline{w} \in C^{\alpha,\alpha/3}(\overline{B}_R \times [0,T])$ we can solve the linearized inhomogeneous Hele-Shaw problem (3.4)–(3.6) with $w = \overline{w}$ and $\rho|_{t=0} = \rho_0$ for $(q,\rho) \in C^{2+\alpha,\alpha/3}(\overline{B}_R \times [0,T]) \times C^{4+\alpha,1+\alpha/3}(\partial B_R \times [0,T])$ and derive the estimate

$$\|\rho\|_{C^{4+\alpha,1+\alpha/3}(\partial B_R \times [0,T])} \le C\{\|\overline{w}\|_{C^{\alpha,\alpha/3}(\overline{B}_R \times [0,T])} + \|\rho_0\|_{C^{4+\alpha}(\partial B_R)}\};$$

the proof is similar to that given for the inhomogeneous Hele-Shaw in [10]. Solving then the system (3.2), (3.3) for w, we get a mapping $S : \overline{w} \to w$ with

$$\begin{aligned}&\|w\|_{C^{2+2\alpha/3,1+\alpha/3}(\overline{B}_R \times [0,T])}\\ &\quad \le C\{\|\overline{w}\|_{C^{\alpha,\alpha/3}(\overline{B}_R \times [0,T])} + \|\rho_0\|_{C^{4+\alpha}(\partial B_R)} + \|w_0\|_{C^{2+\alpha}(\overline{B}_R)}\}.\end{aligned}$$

If w_1 and w_2 correspond to $\overline{w}_1$ and $\overline{w}_2$, then we also have

$$\begin{aligned}\|w_1 - w_2\|_{C^{\alpha,\alpha/3}(\overline{B}_R \times [0,T])} &\le CT^{1-\alpha/3}\|w_1 - w_2\|_{C^{2,1}(\overline{B}_R \times [0,T])}\\ &\le CT^{1-\alpha/3}\|\overline{w}_1 - \overline{w}_2\|_{C^{\alpha,\alpha/3}(\overline{B}_R \times [0,T])},\end{aligned}$$

so that S is a contraction if T is small. Thus S has a unique fixed point, which is the solution of (3.2)–(3.6) for $0 \le t \le T$. This procedure can be extended step-by-step to all $t > 0$. □

The rest of the paper is devoted to the problem: Find the values of $\mu = \mu(R)$ such that the solution of (3.2)–(3.6) satisfies (1.10) for any choice of the initial data (3.7).

We proceed formally to work with the Laplace transforms of w, q and ρ,

$$\begin{aligned}\widehat{w}(r,\theta,\varphi,s) &= \int_0^\infty e^{-st} w(r,\theta,\varphi,t)dt,\\ \widehat{q}(r,\theta,\varphi,s) &= \int_0^\infty e^{-st} q(r,\theta,\varphi,t)dt,\\ \widehat{\rho}(\theta,\varphi,s) &= \int_0^\infty e^{-st} \rho(\theta,\varphi,t)dt,\end{aligned}$$

for $\mathrm{Re}s > a$. Taking the Laplace transform in (3.2)–(3.6), we get

$$\begin{aligned}-\Delta\widehat{w} + (s+1)\widehat{w} &= w(r,\theta,\varphi,0) \quad \text{in } B_R,\\ \widehat{w}(R,\theta,\varphi,s) &= -\lambda\widehat{\rho}(s),\end{aligned} \tag{3.8}$$

$$\begin{aligned}\Delta\widehat{q} &= -\mu\widehat{w} \quad \text{in } B_R,\\ \widehat{q}(R,\theta,\varphi,s) &= -\frac{1}{R^2}\Big(\widehat{\rho} + \frac{1}{2}\Delta_\omega\widehat{\rho}\Big),\end{aligned} \tag{3.9}$$

$$s\widehat{\rho} - \rho(\theta,\varphi,0) = \mu(1-\widetilde{\sigma})\widehat{\rho} - \frac{\partial\widehat{q}}{\partial r}\Big|_{r=R}. \tag{3.10}$$

In particular, if

$$\begin{aligned}&\rho(\theta,\varphi,0) = \rho_0 Y_{n,m}(\theta,\varphi) \quad (\rho_0 \text{ constant}),\\ &w(r,\theta,\varphi,0) = w_0(r)Y_{n,m}(\theta,\varphi)\end{aligned} \tag{3.11}$$

where $Y_{n,m}$ is the spherical harmonic of order (n,m), then we seek solution of the form

$$\begin{aligned}\rho(\theta,\varphi,t)&=\rho_{n,m}(t)Y_{n,m}(\theta,\varphi),\\ w(r,\theta,\varphi,t)&=w_{n,m}(r,t)Y_{n,m}(\theta,\varphi),\\ q(r,\theta,\varphi,t)&=q_{n,m}(r,t)Y_{n,m}(\theta,\varphi).\end{aligned}\tag{3.12}$$

Introducing the Laplace transforms

$$\begin{aligned}\widehat{\rho}&=\widehat{\rho}_{n,m}(s)Y_{n,m}(\theta,\varphi),\\ \widehat{w}&=\widehat{w}_{n,m}(r,s)Y_{n,m}(\theta,\varphi),\\ \widehat{q}&=\widehat{q}_{n,m}(r,s)Y_{n,m}(\theta,\varphi).\end{aligned}\tag{3.13}$$

and using the relation

$$\Delta_\omega Y_{n,m}(\theta,\varphi)+n(n+1)Y_{n,m}(\theta,\varphi)=0,\tag{3.14}$$

we find that

$$\begin{aligned}-\Delta\widehat{w}_{n,m}(r,s)+\Big(\frac{n(n+1)}{r^2}+(s+1)\Big)\widehat{w}_{n,m}(r,s)=w_0(r)\quad&\text{in } B_R,\\ \widehat{w}_{n,m}(R,s)=-\lambda\widehat{\rho}_{n,m}(s),&\end{aligned}\tag{3.15}$$

$$\begin{aligned}\Delta\widehat{q}_{n,m}(r,s)-\frac{n(n+1)}{r^2}\widehat{q}_{n,m}(r,s)=-\mu\widehat{w}_{n,m}(r,s)\quad&\text{in } B_R,\\ \widehat{q}_{n,m}(R,s)=-\frac{1}{R^2}\Big(1-\frac{n(n+1)}{2}\Big)\widehat{\rho}_{n,m}(s),&\end{aligned}\tag{3.16}$$

and

$$s\widehat{\rho}_{n,m}(s)-\rho_0=\mu(1-\widetilde{\sigma})\widehat{\rho}_{n,m}-\frac{\partial\widehat{q}_{n,m}}{\partial r}\Big|_{r=R}.\tag{3.17}$$

We can solve (3.15) in the form

$$\widehat{w}_{n,m}=-\lambda\widehat{\rho}_{n,m}(s)\frac{I_{n+1/2}(r\sqrt{s+1})}{r^{1/2}}\,\frac{R^{1/2}}{I_{n+1/2}(R\sqrt{s+1})}+w_{1n},\tag{3.18}$$

where $w_{1n}(r,s)$ is the solution of

$$\begin{aligned}-\Delta w_{1n}+\Big(\frac{n(n+1)}{r^2}+(s+1)\Big)w_{1n}=w_0(r)\quad&\text{in } B_R,\\ w_{1n}\Big|_{r=R}=0.&\end{aligned}\tag{3.19}$$

The function $\psi=\widehat{q}_{n,m}+\frac{\mu}{s+1}\widehat{w}_{n,m}$ satisfies

$$\begin{aligned}-\Delta\widehat{\psi}+\frac{n(n+1)}{r^2}\psi=\frac{\mu}{s+1}w_0(r)\quad&\text{in } B_R,\\ \psi(R,s)=-\frac{1}{R^2}\Big(1-\frac{n(n+1)}{2}\Big)\widehat{\rho}_{n,m}(s)-\frac{\mu\lambda}{s+1}\widehat{\rho}_{n,m}(s),&\end{aligned}\tag{3.20}$$

and therefore

$$\psi = \frac{r^n}{R^n}\Big[\frac{1}{R^2}\Big(\frac{n(n+1)}{2}-1\Big)-\frac{\mu\lambda}{s+1}\Big]\widehat{\rho}_{n,m}+\psi_{1n}, \tag{3.21}$$

where ψ_{1n} is the solution of

$$\begin{aligned} -\Delta\psi_{1n}+\frac{n(n+1)}{r^2}\psi_{1n} &= \frac{\mu}{s+1}w_0(r) \quad \text{in } B_R,\\ \psi_{1n}\Big|_{r=R} &= 0. \end{aligned} \tag{3.22}$$

Using (3.18), (3.21), we find that

$$\begin{aligned} \widehat{q}_{n,m} &= \frac{r^n}{R^n}\Big[\frac{1}{R^2}\Big(\frac{n(n+1)}{2}-1\Big)-\frac{\mu\lambda}{s+1}\Big]\widehat{\rho}_{n,m}-\frac{\mu}{s+1}\widehat{w}_{n,m}+\psi_{1n}\\ &= \Big\{\frac{r^n}{R^n}\Big[\frac{1}{R^2}\Big(\frac{n(n+1)}{2}-1\Big)-\frac{\mu\lambda}{s+1}\Big]\\ &\qquad +\frac{\mu\lambda}{s+1}\frac{I_{n+1/2}(r\sqrt{s+1})}{r^{1/2}}\,\frac{R^{1/2}}{I_{n+1/2}(R\sqrt{s+1})}\Big\}\widehat{\rho}_{n,m}\\ &\qquad -\frac{\mu}{s+1}w_{1n}+\psi_{1n}, \end{aligned} \tag{3.23}$$

so that

$$\begin{aligned} \frac{\partial\widehat{q}_{n,m}}{\partial r}\Big|_{r=R} &= \Big\{\frac{n}{R}\Big[\frac{1}{R^2}\Big(\frac{n(n+1)}{2}-1\Big)-\frac{\mu\lambda}{s+1}\Big]\\ &\qquad +\frac{\mu\lambda}{s+1}\frac{d}{dr}\Big[\frac{I_{n+1/2}(r\sqrt{s+1})}{r^{1/2}}\Big]_{r=R}\,\frac{R^{1/2}}{I_{n+1/2}(R\sqrt{s+1})}\Big\}\widehat{\rho}_{n,m}\\ &\qquad +\frac{d}{dr}\Big[-\frac{\mu}{s+1}w_{1n}+\psi_{1n}\Big]_{r=R}, \end{aligned} \tag{3.24}$$

and, by (2.7),

$$\frac{d}{dr}\Big[\frac{I_{n+1/2}(r\sqrt{s+1})}{r^{1/2}}\Big] = \frac{n}{r^{3/2}}I_{n+1/2}(r\sqrt{s+1})+\frac{\sqrt{s+1}}{r^{1/2}}I_{n+3/2}(r\sqrt{s+1}). \tag{3.25}$$

We substitute (3.25) into (3.24) and insert the resulting expression for $(\partial\widehat{q}_{n,m}/\partial r)(R)$ into (3.17). We then get

$$\begin{aligned} \Big\{s-\mu(1-\widetilde{\sigma})+\frac{n}{R^3}\Big(\frac{n(n+1)}{2}-1\Big)+\frac{\mu\lambda}{\sqrt{s+1}}\frac{I_{n+3/2}(R\sqrt{s+1})}{I_{n+1/2}(R\sqrt{s+1})}\Big\}\widehat{\rho}_{n,m}\\ = \rho_0-\frac{d}{dr}\Big[-\frac{\mu}{s+1}w_{1n}+\psi_{1n}\Big]_{r=R}. \end{aligned} \tag{3.26}$$

Noting that, by (2.10)–(2.13) and (1.8),

$$\begin{aligned} &\lambda = \sigma_r(R) = \coth R-\frac{1}{R} = \frac{I_{3/2}(R)}{I_{1/2}(R)} = RP_0(R),\\ &\text{and } \frac{1-\widetilde{\sigma}}{\lambda R} = P_1(R), \end{aligned} \tag{3.27}$$

we can rewrite (3.26) as

$$\widehat{\rho}_{n,m} = \frac{1}{\mu R^2 P_0(R) h_n(s,\mu,R)}\Big\{\rho_0 + \frac{d}{dr}\Big[\frac{\mu}{s+1}w_{1n} - \psi_{1n}\Big]_{r=R}\Big\}, \tag{3.28}$$

where h_n is given by

$$\begin{aligned} h_n(s) \equiv h_n(s,\mu,R) = \frac{1}{\mu}\Big[\frac{s}{R^2P_0(R)} + \frac{n}{R^5P_0(R)}\Big(\frac{n(n+1)}{2} - 1\Big)\Big] \\ - P_1(R) + P_n(R\sqrt{s+1}). \end{aligned} \tag{3.29}$$

Thus the behavior of $\rho_{n,m}$ should strongly depend on the properties of h_n.

Lemma 3.2 *Let $n \geq 0$ be an integer and consider the elliptic problem*

$$-\Delta w + \frac{n(n+1)}{r^2}w = b(r,s) \quad \text{in } B_R, \tag{3.30}$$

$$w\Big|_{r=R} = 0, \tag{3.31}$$

where s is any complex number.

(i) If for some s, $b(\cdot,s) \in L^2(B_R)$, then this problem has a unique solution w in $H^2(B_R)$; furthermore,

$$\begin{aligned} \|w\|_{H^2(B_R)} + n\Big[\int_0^R |w_r(r,s)|^2 dr\Big]^{1/2} + n^2\Big[\int_0^R \Big|\frac{w(r,s)}{r}\Big|^2 dr\Big]^{1/2} \\ \leq C\Big[\int_0^R |b(r,s)|^2 r^2 dr\Big]^{1/2}, \end{aligned} \tag{3.32}$$

$$\Big|\frac{\partial w}{\partial r}(R,s)\Big| \leq \frac{C}{(n+1)^{1/2}}\ \Big[\int_0^R |b(r,s)|^2 r^2 dr\Big]^{1/2}, \tag{3.33}$$

where C is a constant independent of s and n.

(ii) If $b(r,s)$ is holomorphic in s in some region and $b(\cdot,s) \in L^2(B_R)$, then $w(r,s)$ is holomorphic in s in the same region, and

$$\Big\|\frac{\partial w}{\partial s}\Big\|_{H^2(B_R)} + n\Big[\int_0^R \Big|\frac{\partial^2 w}{\partial s \partial r}\Big|^2 dr\Big]^{1/2} \leq C\Big[\int_0^R \Big|\frac{\partial b(r,s)}{\partial s}\Big|^2 r^2 dr\Big]^{1/2}, \tag{3.34}$$

where C is a constant independent of s and n.

Proof. If $n = 0$, then the existence, uniqueness as well as (3.32)–(3.34) follow from the standard L^2 theory for elliptic equations.

If $n \geq 1$, we consider first the equation (3.30) in $\delta < r < R$ with zero boundary values on $x = R$ and $x = \delta$, where δ is positive and arbitrarily small. Since we can solve separately the real part and imaginary part of

$b(r,s)$, we may assume without loss of generality that $b(r,s)$ is real valued. Multiplying (3.30) with w/r^2 and integrating over $B_R \setminus B_\delta$, we obtain

$$\begin{aligned}
&\int_\delta^R \Big|\frac{\partial w}{\partial r}\Big|^2 dr + n(n+1)\int_\delta^R \Big|\frac{w}{r}\Big|^2 dr \\
&= \int_\delta^R b(r,s)w(r,s)dr + \int_\delta^R \frac{2w}{r}\frac{\partial w}{\partial r}dr \\
&\le \frac{n^2+n-1}{2}\int_\delta^R \Big|\frac{w}{r}\Big|^2 dr + \frac{1}{2(n^2+n-1)}\int_\delta^R |b(r,s)|^2 r^2 dr \\
&\qquad + \int_\delta^R \Big|\frac{\partial w}{\partial r}\Big|^2 dr + \int_\delta^R \Big|\frac{w}{r}\Big|^2 dr,
\end{aligned}$$

from which it follows that

$$\int_\delta^R \Big|\frac{w}{r}\Big|^2 dr \le \frac{1}{(n^2+n-1)^2}\int_\delta^R |b(r,s)|^2 r^2 dr. \tag{3.35}$$

Similarly, multiplying the equation (3.30) by Δw and integrating over $B_R \setminus B_\delta$, we obtain

$$\begin{aligned}
&\int_\delta^R |\Delta w|^2 r^2 dr + n(n+1)\int_\delta^R \Big|\frac{\partial w}{\partial r}\Big|^2 dr \\
&= -\int_\delta^R \Delta w(r,s)b(r,s)r^2 dr + 2n(n+1)\int_\delta^R \frac{w}{r}\,\frac{\partial w}{\partial r}dr \\
&\le \frac{1}{2}\int_\delta^R |\Delta w|^2 r^2 dr + \frac{1}{2}\int_\delta^R |b(r,s)|^2 r^2 dr \\
&\qquad + \frac{1}{2}n(n+1)\int_\delta^R \Big|\frac{\partial w}{\partial r}\Big|^2 dr + 2n(n+1)\int_\delta^R \Big|\frac{w}{r}\Big|^2 dr,
\end{aligned}$$

which implies, using (3.35), that

$$\int_\delta^R |\Delta w|^2 r^2 dr + n^2\int_\delta^R \Big|\frac{\partial w}{\partial r}\Big|^2 dr \le C\int_\delta^R |b(r,s)|^2 r^2 dr, \tag{3.36}$$

where C is independent of n and δ. Letting $\delta \to 0$ we obtain a solution to (3.30)–(3.31). Since $w = 0$ on ∂B_R, we have$\|w\|_{H^2(B_R)} \le C\|\Delta w\|_{L^2(B_R)}$, so that (3.36) implies (3.32).

Uniqueness of solutions w in $H^2(B_R)$ follows by taking $b \equiv 0$ and repeating the derivation of (3.32) with $\delta = 0$.

To prove (3.33), we note that by mean value theorem,

$$\int_{R-R/n}^R \Big|\frac{\partial w}{\partial r}\Big|^2 dr = \frac{R}{n}\Big|\frac{\partial w(r_1)}{\partial r}\Big|^2 \text{ for some } r_1 \in [R - R/n, R],$$

and using (3.32) we then get

$$\Big|\frac{\partial w(r_1)}{\partial r}\Big| \le \frac{C}{n^{1/2}}\Big[\int_0^R |b(r,s)|^2 r^2 dr\Big]^{1/2}. \tag{3.37}$$

Integrating (3.30) over $B_R \setminus B_{R-R/n}$, using (3.35) and (3.37), we obtain

$$\begin{aligned} R^2\Big|\frac{\partial w}{\partial r}(R)\Big| &\le R^2(1-\frac{1}{n})\Big|\frac{\partial w}{\partial r}(r_1)\Big| + \int_{r_1}^R n(n+1)|w|dr + \int_{r_1}^R |b(r,s)|r^2dr \\ &\le R^2(1-\frac{1}{n})\Big|\frac{\partial w}{\partial r}(r_1)\Big| + (R-r_1)^{1/2}n(n+1)\Big[\int_{r_1}^R |w|^2dr\Big]^{1/2} \\ &\quad + \frac{(R^3-r_1^3)^{1/2}}{3}\Big[\int_{r_1}^R |b(r,s)|^2r^2dr\Big]^{1/2} \\ &\le \frac{C}{n^{1/2}}\Big[\int_0^R |b(r,s)|^2r^2dr\Big]^{1/2}. \end{aligned}$$

To prove (3.34) consider the equation for $W(r,s,\Delta s) \equiv [w(r,s+\Delta s) - w(r,s)]/\Delta s$ and apply to it the estimate (3.32). We find that $W(r,s,\Delta s)$ is uniformly bounded in r and, as $\Delta s \to 0$, it converges to the (unique) solution of

$$\begin{aligned} &-\Delta V + \frac{n(n+1)}{r^2}V = \frac{\partial b(r,s)}{\partial s} \quad \text{in } B_R, \\ &V\Big|_{r=R} = 0. \end{aligned}$$

This shows that $w(r,s)$ is holomorphic in s. Finally, applying (3.32) to V we obtain the estimate (3.34). □

In the next lemma we consider a more complicated situation in which the elliptic operator depends on s in a special way and s is restricted to a half plane.

Lemma 3.3 *Let $n \ge 0$ and consider the elliptic problem.*

$$-\Delta w + \Big(\frac{n(n+1)}{r^2} + (s+1)\Big)w = b(r,s) \quad \text{in } B_R, \tag{3.38}$$

$$w\Big|_{r=R} = 0, \tag{3.39}$$

where s is any complex number with $\mathrm{Re}\, s \ge -\delta_0(1+n)$ (where δ_0 is positive and small) and $b \in L^2(B_R)$. Then for $\mathrm{Re}\, s \ge -\delta_0(1+n)$, this problem has a unique solution in $H^2(B_R)$ as asserted in Lemma 3.2, and the inequalities (3.32), (3.33) hold; furthermore, if $b(r,s)$ is holomorphic in $\mathrm{Re}\, s \ge -\delta_0(1+n)$, also the inequality

$$\Big\|\frac{\partial w}{\partial s}\Big\|_{H^2(B_R)} + n\Big[\int_0^R \Big|\frac{\partial^2 w}{\partial s \partial r}\Big|^2 dr\Big]^{1/2} \le C\Big[\int_0^R \Big\{\Big|\frac{\partial b(r,s)}{\partial s}\Big|^2 + |b(r,s)|^2\Big\}r^2dr\Big]^{1/2}, \tag{3.40}$$

holds, and the constants C in (3.32)–(3.33) and (3.40) are independent of s and n.

Proof. The existence and uniqueness is the same as the previous lemma. Writing $s = s_1 + is_2$, $w = u + iv$, we obtain

$$-\Delta u + (\frac{n(n+1)}{r^2} + s_1 + 1)u - s_2 v = \text{ Re } b,$$
$$-\Delta v + (\frac{n(n+1)}{r^2} + s_1 + 1)v + s_2 u = \text{ Im } b.$$

Multiplying the first equation by Δu and second equation by Δv and integrating over B_R, we obtain

$$\begin{aligned}
&\int_0^R [(\Delta u)^2 + (\Delta v)^2]r^2 dr + \int_0^R \{n(n+1) + r^2(s_1+1)\}(|u_r|^2 + |v_r|^2)dr \\
&= \int_0^R 2n(n+1)\frac{uu_r + vv_r}{r}dr - \int_0^R [\text{ Re } b \cdot \Delta u + \text{ Im } b \cdot \Delta v]r^2 dr \\
&\le \frac{1}{2}n(n+1)\int_0^R (|u_r|^2 + |v_r|^2)dr + 2n(n+1)\int_0^R \frac{u^2+v^2}{r^2}dr \\
&\qquad + \frac{1}{2}\int_0^R [(\Delta u)^2 + (\Delta v)^2]r^2 dr + \frac{1}{2}\int_0^R |b(r,s)|^2 r^2 dr,
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{1}{2}\int_0^R [(\Delta u)^2 + (\Delta v)^2]r^2 dr + \int_0^R \{\frac{n(n+1)}{2} + r^2(s_1+1)\}(|u_r|^2 + |v_r|^2)dr \\
&\qquad \le \frac{1}{2}\int_0^R |b(r,s)|^2 r^2 dr + 2n(n+1)\int_0^R \frac{u^2+v^2}{r^2}dr.
\end{aligned} \tag{3.41}$$

In case $n = 0$, (3.41) implies implies (3.32) and (by Sobolev's inequalities) (3.33). In case $n \ge 1$, we need to further estimate the second term on the right-hand side of (3.41).

Clearly $|w|^2 = (u^2 + v^2)$ satisfies

$$-\frac{1}{2}\Delta(|w|^2) + (\frac{n(n+1)}{r^2} + s_1 + 1)|w|^2 = (u \text{ Re } b + v \text{ Im } b) - (|\nabla u|^2 + |\nabla v|^2).$$

Since $w(R) = w_r(R) = 0$, we can integrate this equality to obtain

$$\begin{aligned}
&\int_0^R (|u_r|^2 + |v_r|^2)dr + \int_0^R (\frac{n(n+1)}{r^2} + s_1 + 1)|w|^2 dr \\
&= \int_0^R \frac{2uu_r + 2vv_r}{r}dr + \int_0^R (u \text{ Re } b + v \text{ Im } b)dr,
\end{aligned}$$

from which we obtain

$$\int_0^R \Big|(u^2+v^2)\Big(\frac{n^2+n-1}{r^2} + s_1 + 1\Big)\Big|dr \le \frac{1}{(n^2+n-1)}\int_0^R |b(r,s)|^2 r^2 dr. \tag{3.42}$$

provided $s_1 > -\delta_0(n+1)$ and δ_0 is small. Hence

$$\int_0^R \Big|\frac{u^2+v^2}{r^2}\Big| dr \le \frac{C}{n^4}\int_0^R |b(r,s)|^2 r^2 dr. \tag{3.43}$$

Substituting (3.43) into (3.41), we obtain (3.32). Using (3.42) we can proceed to derive the estimate (3.33) as in Lemma 3.2.

Consider next the equation for $W(r,s,\Delta s) = [w(r,s+\Delta s) - w(r,s)]/\Delta s$ and apply to it the estimate (3.32). We find that $W(r,s,\Delta s)$ is uniformly bounded in r and, as $\Delta s \to 0$, it converges to the solution of

$$\begin{aligned} &-\Delta V + (\frac{n(n+1)}{r^2} + s + 1)V = \frac{\partial b(r,s)}{\partial s} - w \quad \text{ in } B_R, \\ &V\Big|_{r=R} = 0. \end{aligned}$$

Applying (3.32) to V we obtain (3.40). □

4. The zeros of $h_n(s,\mu,R)$

Lemma 4.1 *For any $\mu > 0$ and any $n \ge 0$, all zeros of $h_n(s) \equiv h_n(s,\mu,R)$, except at most two, are real and less than -1.*

Proof. From the formula (2.19) it is clear that the function $h_n(s,\mu,R)$ $(n \ge 0)$ is holomorphic in $\mathbb{C}$ except at the sequence of points

$$s = -1 - \Big(\frac{j_{n+1/2,m}}{R}\Big)^2, \quad m = 1,2,3,\cdots, \tag{4.1}$$

where it has simple poles.

We introduce the functions

$$\begin{aligned} k_{1n}(s) &= \frac{1}{\mu}\Big[\frac{s}{R^2 P_0(R)} + \frac{n}{R^5 P_0(R)}\Big(\frac{n(n+1)}{2} - 1\Big)\Big], \\ k_{2n}(s) &= P_n(R\sqrt{s+1}) - P_1(R), \end{aligned} \tag{4.2}$$

so that

$$h_n(s) \equiv h_n(s,\mu,R) = k_{1n}(s) + k_{2n}(s). \tag{4.3}$$

As easily seen from (2.19) and (2.18), the function $k_{2n}(s)$ is uniformly bounded in $\mathbb{C}$ outside any small ε_0-neighborhood of the sequence (4.1). Furthermore, we can choose arbitrarily large numbers A such that each sphere

$$\partial B_A = \{r = A\}$$

avoids the ε_0-neighborhood of the sequence (4.1). But then

$$|h_n(s) - k_{1n}(s)| = |k_{2n}(s)| \le \text{const.} < |k_{1n}(s)|$$

on ∂B_A if A is large enough. Hence, inside B_A,

$$\begin{aligned}&\text{(the number of zeros} - \text{ the number of poles) of } h_n(s)\\ &= \text{(the number of zeros} - \text{ the number of poles) of } k_{1n}(s) \ = \ 1.\end{aligned} \tag{4.4}$$

Set

$$\beta_m \equiv -1 - \Big(\frac{j_{n+1/2,m}}{R}\Big)^2$$

and note that in any interval $\beta_{m+1} < s < \beta_m$,

$$h_n(\beta_{m+1} + 0) = +\infty, \qquad h_n(\beta_m - 0) = -\infty.$$

It follows that $h_n(s)$ has at least one zero, say $s = \alpha_{m+1}$, in the interval (β_{m+1}, β_m).

Recalling (4.4) we conclude that $h(s)$ has precisely two more zeros, say α_1 and α_0, which may either be both real, or one is the complex conjugate of the other. □

Setting

$$\alpha_1 = \alpha_1(\mu) \text{ and } \alpha_0 = \alpha_0(\mu)$$

we have, for any $\mu > 0$, either $\text{Im } \alpha_1 = \text{Im } \alpha_0 = 0$ and $\alpha_1 \le \alpha_0$, or $\text{Im } \alpha_1 > 0$, $\alpha_1 = \overline{\alpha}_0$.

Lemma 4.2 *For $n \ge 2$,*

$$h_n(0, \mu_n(R), R) = 0, \tag{4.5}$$

$$h_n(0, \mu, R) > 0 \quad \textit{for } \mu < \mu_n(R), \tag{4.6}$$

$$h_n(0, \mu, R) < 0 \quad \textit{for } \mu > \mu_n(R). \tag{4.7}$$

Proof. Solving the equation $h_n(0, \mu, R) = 0$, we obtain

$$\mu = \frac{n[n(n+1) - 2]}{2R^5 P_0(R)[P_1(R) - P_n(R)]} = \mu_n(R), \tag{4.8}$$

where the last equality is obtained from the definition of P_0, P_1 and P_n. The inequalities (4.6) and (4.7) follows immediately from the form of h_n. □

Lemma 4.3 *(i) For $n \ge 2$, if $\mu > \mu_n(R)$, then $h_n(s, \mu, R)$ has a positive real zero.*

(ii) For $n = 0$, if μ is sufficiently large, then $h_0(s, \mu, R)$ has two positive real zeros.

Proof. If $n \ge 2$ and $\mu > \mu_n(R)$, then $h_n(0, \mu, R) < 0$. But since $h_n(+\infty, \mu, R) = +\infty$, $h_n(s, \mu, R)$ must have a real positive zero.

If $n = 0$ then, by (3.29) and (2.25),

$$h_0(0, \mu, R) = P_0(R) - P_1(R) > 0.$$

Notice that $P_0(r) \to 0$ as $r \to \infty$. Hence if $\overline{s} \gg 1$, then $P_0(R\sqrt{\overline{s} + 1}) - P_1(R) < 0$, and so

$$h_0(\overline{s}, \mu, R) < 0 \quad \text{ if } \mu/\overline{s} \gg 1.$$

Since also $h(+\infty, \mu, R) = +\infty$, $h_n(s, \mu, R)$ has two real positive zeros. □

Lemma 4.4 *If $n = 0$, or $n \geq 2$, and μ sufficiently small, then all zeros of $h_n(s, \mu, R)$ have negative real parts.*

Proof. As before,

$$h_0(0, \mu, R) > 0. \tag{4.9}$$

By (2.11),

$$h_0(\beta_1 + 0) = +\infty \quad \text{where } \beta_1 = -1 - \left(\frac{\pi}{R}\right)^2,$$

Also,

$$h_0(-1, \mu) < 0$$

if μ is sufficiently small, so that $h_0(s, \mu, R)$ must have two real zeros in the interval $\beta_1 < s < 0$:

$$\beta_1 < \alpha_1(\mu) < \alpha_0(\mu) < 0 \quad \text{if } \mu \text{ is small.} \tag{4.10}$$

Thus all the zeros of $h_0(s)$ are accounted for and they are all real and negative.

In case $n \geq 2$, we can argue as in the proof of Lemma 4.1 with B_A replaced by $D_A = \{s; |s| < A, \text{ Re } s < 0\}$. The same argument shows that, for $0 < \mu \ll 1$, the right-hand side of (4.4) is 1, and that $\alpha_1(\mu)$ and $\alpha_0(\mu)$ must lie in D. □

It remains to identify the location in $\mathbb{C}$ of the zeros $\alpha_1(\mu)$, $\alpha_0(\mu)$ for intermediate values of μ.

As μ increases from small to large values there must occur a first value $\mu_n^*(R)$ such that the largest of the real part of the roots of $h_n(s, \mu, R)$ vanishes at $\mu = \mu_n^*(R)$, i.e., Re $\alpha_0(\mu) < 0$ if $\mu < \mu_n^*(R)$ and Re $\alpha_0(\mu_n^*(R)) = 0$.

Lemma 4.5 *For $n = 0$ or $n \geq 2$, if $\mu > \mu_n^*(R)$ then at least one zero of $h_n(s, \mu, R)$ lies in $\{Re\ (s) > 0\}$, i.e., Re $\alpha_0(\mu) > 0$.*

To prove the lemma, we need to show that the imaginary axis can only be crossed once by the root $\alpha_0(\mu)$ of $h_n(s, \mu, R)$. At such a crossing $\alpha_0(\mu) = i\tau$, $\tau \geq 0$ and

$$h_n(i\tau, \mu, R) = 0, \quad \tau \geq 0, \tag{4.11}$$

that is,

$$\frac{n[n(n+1)/2 - 1]}{\mu R^5 P_0(R)} - P_1(R) + \text{Re } P_n(R\sqrt{i\tau + 1}\,) = 0, \tag{4.12}$$

$$\frac{\tau}{\mu R^2 P_0(R)} + \text{Im } P_n(R\sqrt{i\tau + 1}\,) = 0. \tag{4.13}$$

We need to show that the last two equations have at most one solution pair (τ, μ). In the case $\tau > 0$, by eliminating μ from the second equation, we obtain from the first equation

$$f_n(\tau, R) = 0, \quad \tau > 0 \quad (R \text{ fixed}), \tag{4.14}$$

where

$$f_n(\tau,R)\equiv P_1(R)-\text{Re }P_n(R\sqrt{i\tau+1}\,)+\frac{n[n(n+1)-2]}{2R^3}\text{Im }\frac{P_n(R\sqrt{i\tau+1}\,)}{\tau}. \tag{4.15}$$

Lemma 4.6 *For $n=0$ or $n\geq 2$,*

$$\frac{d}{d\tau}f_n(\tau,R)>0 \text{ for all } \tau>0. \tag{4.16}$$

Proof. By (2.18),

$$\begin{aligned} P_n(R\sqrt{i\tau+1}) &= 2\sum_{m=1}^{\infty}\frac{1}{R^2(i\tau+1)+j_{n+1/2,m}^2} \\ &= 2\sum_{m=1}^{\infty}\frac{(R^2+j_{n+1/2,m}^2)-iR^2\tau}{R^4\tau^2+(R^2+j_{n+1/2,m}^2)^2}. \end{aligned} \tag{4.17}$$

Thus, for $\tau>0$,

$$\frac{d}{d\tau}\text{Re }P_n(R\sqrt{i\tau+1})=2\sum_{m=1}^{\infty}\frac{-2R^4\tau(R^2+j_{n+1/2,m}^2)}{[R^4\tau^2+(R^2+j_{n+1/2,m}^2)^2]^2}<0, \tag{4.18}$$

$$\frac{d}{d\tau}\text{Im }\frac{P_n(R\sqrt{i\tau+1})}{\tau}=2\sum_{m=1}^{\infty}\frac{2R^6\tau}{[R^4\tau^2+(R^2+j_{n+1/2,m}^2)^2]^2}>0, \tag{4.19}$$

which immediately imply that $\frac{d}{d\tau}f_l(\tau,R)>0$ for all $\tau>0$. □

Proof of Lemma 4.5. From (4.15) and Lemma 4.6 it follows that $f_n(\tau,R)$ vanishes in at most one point τ. Since $f_n(+\infty,\mu,R)=P_1(R)>0$, as seen from (4.16)

(4.14) has exactly one solution at $\tau>0$ if $f_n(0,R)<0$,

(4.14) has exactly one solution at $\tau=0$ if $f_n(0,R)=0$,

(4.14) has no solution for $\tau\geq 0$ if $f_n(0,R)>0$.

In the case $n=0$, $f_0(0,R)=P_1(R)-P_0(R)<0$, so that the equation (4.14) determines τ uniquely, and then the equation (4.12) determines $\mu_0^*(R)$ uniquely.

Consider the case $n\geq 2$. If $f_n(0,R)<0$, then (4.14) uniquely determines τ, and again (4.12) determines μ_n^*; in this case $\alpha_0(\mu_0^*)=i\tau$ and $\alpha_1(\mu_0^*)=-i\tau$. On the other hand if $f_n(0,R)\geq 0$ then, by Lemma 4.6, $f_n(\tau,R)>0$ if $\tau>0$. It follows that $\alpha_0(\mu_n^*)=0$. Since by Lemma 4.2, $\mu=\mu_n(R)$ is the unique zero of $h_n(0,\mu,R)$, we must have $\mu_n^*(R)=\mu_n(R)$. □

Remark 4.1 From Lemma 4.5 it follows that $\mu_n^*(R)$ is equal to $\mu_n(R)$ if $\alpha_0(\mu_n^*(R))$ is real, and $\mu_n^*(R)<\mu_n(R)$ if $\alpha_0(\mu_n^*)$ is a complex number. Hence

$$\mu_n^*(R)\leq\mu_n(R).$$

We shall later determine for which R the equality holds.

A direct computation shows that for $n \geq 2$,

$$\begin{aligned}\frac{\partial h_n}{\partial s}(0, \mu_n(R), R) &= \frac{1}{\mu_n(R) R^2 P_0(R)} + \frac{R}{2} P_n'(R) \\ &= \frac{R^3}{n[n(n+1)/2 - 1]} \{P_1(R) - P_n(R)\} + \frac{R}{2} P_n'(R),\end{aligned} \tag{4.20}$$

so that, by (2.13), (2.14),

$$\lim_{R \to 0} \frac{1}{R^2} \frac{\partial h_n}{\partial s}(0, \mu_n(R), R) = \frac{-1}{(2n+3)^2(2n+5)} < 0, \tag{4.21}$$

$$\lim_{R \to +\infty} \frac{1}{R} \frac{\partial h_n}{\partial s}(0, \mu_n(R), R) = \frac{2}{n(n+2)} > 0. \tag{4.22}$$

Lemma 4.7 *For any $n \geq 2$ there exists a unique positive number R_n^* such that*

$$\frac{\partial h_n}{\partial s}(0, \mu_2(R), R) > 0 \quad \text{for } R > R_n^*, \tag{4.23}$$

$$\frac{\partial h_n}{\partial s}(0, \mu_2(R), R) < 0 \quad \text{for } R < R_n^*; \tag{4.24}$$

furthermore,

$$R_n^* < 0.32 \quad \text{for all } n \geq 2. \tag{4.25}$$

Proof. Consider the function

$$b_n(R) \equiv \frac{1}{R^2} \frac{\partial h_n}{\partial s}(0, \mu_n(R), R).$$

By (4.21), $b_n(R) < 0$ for R small and, by (4.22), $b_n(R) > 0$ for R large. Recalling (4.20), we have

$$\frac{d}{dR} b_n(R) = \frac{[P_1(R) + RP_1'(R)] - [P_n(R) + RP_n'(R)]}{n[n(n+1)/2 - 1]} + \frac{1}{2}\Big(\frac{1}{R} P_n'(R)\Big)'. \tag{4.26}$$

From (2.19) we get

$$\Big(\frac{1}{R} P_n'(R)\Big)' = 2 \sum_{m=1}^{\infty} \frac{8R}{(R^2 + j_{n+1/2,m}^2)^3} > 0 \tag{4.27}$$

and

$$P_n(R) + RP_n'(R) = 2 \sum_{m=1}^{\infty} \frac{-R^2 + j_{n+1/2,m}^2}{(R^2 + j_{n+1/2,m}^2)^2}. \tag{4.28}$$

The function

$$c(x) \equiv \frac{-R^2 + x}{(R^2 + x)^2}$$

satisfies

$$\frac{d}{dx}c(x) = \frac{-x + 3R^2}{(R^2 + x)^3} < 0 \quad \text{for } R^2 < \frac{x}{3}.$$

Since $j_{n+1/2,m}$ is monotone increasing in n, we then have

$$\frac{-R^2 + j_{3/2,m}^2}{(R^2 + j_{3/2,m}^2)^2} > \frac{-R^2 + j_{n+1/2,m}^2}{(R^2 + j_{n+1/2,m}^2)^2} \quad \text{if } R^2 < \frac{j_{3/2,m}^2}{3}, \quad n \geq 2.$$

Summing over m and recalling (4.28), we obtain,

$$P_1(R) + RP_1'(R) > P_n(R) + RP_n'(R) \quad \text{for } R < \frac{j_{3/2,1}}{\sqrt{3}} \approx 2.594. \tag{4.29}$$

Combining this estimate with (4.27), we obtain

$$\frac{d}{dR}b_n(R) > 0 \quad \text{for } R < 2.59, \quad n \geq 2. \tag{4.30}$$

By (2.19) and (2.13),

$$\begin{aligned}\frac{1}{2R}P_n'(R) &= 2\sum_{m=1}^{\infty} \frac{-1}{(R^2 + j_{n+1/2,m}^2)^2} \\ &> 2\sum_{m=1}^{\infty} \frac{-1}{j_{n+1/2,m}^4} = \lim_{R\to 0}\frac{1}{2R}P_n'(R) = \frac{-1}{(2n+3)^2(2n+5)},\end{aligned}$$

so that

$$\begin{aligned}b_n(R) &> \frac{2R}{n(n+2)(n-1)}\{P_1(R) - P_n(R)\} - \frac{1}{(2n+3)^2(2n+5)} \\ &> \frac{2}{n(n+2)(n-1)}\Big(RP_1(R) - \frac{R}{2n+3}\Big) - \frac{1}{(2n+3)^2(2n+5)}.\end{aligned} \tag{4.31}$$

where in the last inequality we made use of (2.21). In particular,

$$\begin{aligned}b_n(0.32) &> \frac{2}{n(n+2)(n-1)}\Big(0.0638 - \frac{0.32}{2n+3}\Big) - \frac{1}{(2n+3)^2(2n+5)}. \\ &= \frac{2}{n(n+2)(n-1)}\Big(\frac{0.1276n - 0.1286}{2n+3}\Big) - \frac{1}{(2n+3)^2(2n+5)} \\ &> \frac{2}{n(n+2)(n-1)}\Big(\frac{0.125(n-1)}{2n+3}\Big) - \frac{1}{(2n+3)^2(2n+5)} \\ &= \frac{1}{2n+3}\Big\{\frac{1}{4n(n+2)} - \frac{1}{(2n+3)(2n+5)}\Big\} > 0.\end{aligned}$$

Recalling (4.30) we conclude that $b_n(R) = 0$ has exactly one zero in the interval $[0, 2.59]$, and this zero lies in the interval $(0, 0.32)$.

We next want to show that $b_n(R)$ remains positive for all $R > 2.59$. Since (by Lemma 2.2) $RP_1(R)$ is monotone increasing and $P_n(R) < 1/(2n+3)$, we deduce from (4.31) that

$$\begin{aligned} b_n(R) &> \frac{2}{n(n+2)(n-1)}\Big\{\Big(a_1P_1(a_1) - \frac{a_2}{2n+3}\Big) - \frac{n(n+2)(n-1)}{2(2n+3)^2(2n+5)}\Big\} \\ &> 0 \qquad\qquad \text{for } a_1 < R \le a_2, \end{aligned}$$

provided we choose a_2 such that

$$a_2 \le (2n+3)\Big\{a_1P_1(a_1) - \frac{n(n+2)(n-1)}{2(2n+3)^2(2n+5)}\Big\} \equiv \mathcal{Z}(a_1, n). \tag{4.32}$$

If $a_1 = 2.59$, then $a_1P_1(a_1) \approx 0.4411$, and we can choose

$$a_2 = \min_{n\ge 2} \mathcal{Z}(2.59, n) \ge 3.046.$$

Similarly, with $a_2 = 3.046$, $b_n(R) > 0$ in the interval $(a_2, a_3]$, $(a_3, a_4]$, $(a_4, a_5]$ where

$$\begin{aligned} a_3 &= \min_{n\ge 2} \mathcal{Z}(3.04, n) \ge 3.422, \\ a_4 &= \min_{n\ge 2} \mathcal{Z}(3.42, n) \ge 3.697, \\ a_5 &= \min_{n\ge 2} \mathcal{Z}(3.69, n) \ge 3.873. \end{aligned}$$

Since

$$\Big\{3.87P_1(3.87) - \frac{n(n+2)(n-1)}{2(2n+3)^2(2n+5)}\Big\} > 0.57 - \frac{1}{16} = 0.5075,$$

we can choose $a_6 = 0.5(2n+3)$ so that

$$b_n(R) > 0 \quad \text{for } 2.59 < R \le n + 3/2, \quad n \ge 2.$$

In order to extend this inequality to all $R > 0$, we begin by noting, by (2.19) and (2.17), that

$$\begin{aligned} \frac{-1}{2R}P_n'(R) &= 2\sum_{m=1}^{\infty} \frac{1}{(R^2 + j_{n+1/2,m}^2)^2} \\ &\le \frac{1}{R^2 + j_{n+1/2,1}^2} \sum_{m=1}^{\infty} \frac{2}{R^2 + j_{n+1/2,m}^2} \\ &< \frac{1}{R^2 + (n+1/2)(n+5/2)} P_n(R). \end{aligned} \tag{4.33}$$

Recalling the definition of b_n and using (4.20), we get

$$\begin{aligned} &n(n^2+n-2)b_n(R) \\ &> 2RP_1(R) - \Big(2R + \frac{n(n-1)(n+2)}{R^2 + (n+1/2)(n+5/2)}\Big)P_n(R) \equiv J_n. \end{aligned}$$

Using Lemma 2.3 we find that if $R > n+1$, then

$$\begin{aligned} J_n &> 2R\Big(\frac{1}{R} - \frac{2}{R^2}\Big) \\ &\quad -\Big(2R + \frac{n(n-1)(n+2)}{(n+1)^2 + (n+1/2)(n+5/2)}\Big)\Big(\frac{1}{R} - \frac{n+1}{R^2} + \frac{n^2+n+1}{2R^3}\Big) \\ &> \Big(2 - \frac{4}{R}\Big) - \Big(2R + \frac{n}{2}\Big)\Big(\frac{1}{R} - \frac{n+1}{R^2} + \frac{n^2+n+1}{2R^3}\Big) \\ &= \frac{1}{R}\Big\{\Big(\frac{3n}{2} - 2\Big) - \frac{n^2+n+2}{2R} - \frac{n(n^2+n+1)}{4R^2}\Big\}. \end{aligned}$$

Hence, if $R > n + 3/2$,

$$J_n > \frac{1}{R}\Big\{\Big(\frac{3n}{2} - 2\Big) - \frac{n^2+n+2}{2(n+3/2)} - \frac{n(n^2+n+1)}{4(n+3/2)^2}\Big\} \;>\; 0 \quad \text{for } n \geq 3.$$

It remains to prove that $b_n(R) > 0$ for $n = 2$; this was already proved for $2.59 < R < a_5$, where $a_5 > 3.873$. We have $j_{2\frac{1}{2},1} \approx 5.763$, so that

$$\frac{-1}{2R}P_n'(R) < \frac{1}{R^2 + j_{n+1/2,1}^2}P_n(R) < \frac{1}{R^2+33}P_n(R).$$

Replacing (4.33) by this sharper inequality and proceeding as before, we obtain

$$J_2 > \Big(2 - \frac{4}{R}\Big) - \Big(2R + \frac{8}{R^2+33}\Big)\Big(\frac{1}{R} - \frac{3}{R^2} + \frac{7}{2R^3}\Big) > 0$$

if $R > 3.87$. We conclude that

$$b_n(R) > 0 \quad \text{for } 2.59 < R < \infty, \quad n \geq 2. \tag{4.34}$$

In conclusion, $b_n(R) = 0$ has exactly one zero in $(0, \infty)$, and this zero lies in the interval $(0, 0.32)$. □

We can solve numerically for R_n^*: $R_2^* \approx 0.15876$. Numerical computation shows that R_n^* increases with n, $R_3^* \approx 0.18951$, $R_4^* \approx 0.20999$, $\cdots$, $R_{50}^* \approx 0.30131$, $\cdots$, $R_{100}^* \approx 0.30722$, etc.

Theorem 4.8 *For $n \geq 2$, the following inequalities hold:*

$$\mu_n^*(R) < \mu_n(R) \quad \text{if } R < R_n^*, \tag{4.35}$$
$$\mu_n^*(R) = \mu_n(R) \quad \text{if } R > R_n^*, \tag{4.36}$$

where R_n^* is defined in the previous lemma.

Proof. Since $h_n(0, \mu_n(R), R) = 0$, we must have $\mu_n^*(R) \le \mu_n(R)$.

If $R < R_n^*$, then $(d/ds)h_n(0, \mu_n(R), R) < 0$; therefore $h_n(\overline{s}, \mu_n(R), R) < 0$ for some small $\overline{s} > 0$, and thus for $0 < \mu_n(R) - \mu \ll 1$, $h_n(\overline{s}, \mu, R) < 0$. Since $h_n(+\infty, \mu, R) = +\infty$, $h_n(s, \mu, R)$ must have a positive root so that $\mu_n^*(R) < \mu_n(R)$.

If $R > R_n^*$, then $(d/ds)h(0, \mu_n(R), R) > 0$; therefore $h_n(-\overline{s}, \mu_n(R), R) < 0$ for some small $\overline{s} > 0$, and thus for $0 < \mu_n(R) - \mu \ll 1$, $h_n(-\overline{s}, \mu, R) < 0$. Since $h_n(0, \mu, R) > h_n(0, \mu_n(R), R) = 0$, $h_n(s, \mu, R)$ has two more real negative zeros in addition to the zeros α_m, which lie in $\beta_{m+1}, \beta_m)$, $m \ge 1$, so that $\alpha_0(\mu) < 0$.

Thus if $0 < \mu_2(R) - \mu \ll 1$, all zeros of $h_n(s, \mu, R)$ are real and negative while at $\mu = \mu_n(R)$ $h_n(s, \mu_n(R), R)$ has the root $s = 0$. Since $\alpha_0(\mu)$ cannot cross the imaginary axis more than once, we conclude that $\mu_n^*(R) = \mu_n(R)$. □

Lemma 4.9 *The following inequality holds:*

$$\mu_n^*(R) > \mu_{n-1}(R) \quad \text{for } n \ge 3. \tag{4.37}$$

Proof. If $\mu_n^*(R) = \mu_n(R)$, then clearly

$$\mu_n^*(R) = \mu_n(R) > \mu_{n-1}(R).$$

Thus we only need to consider the case $\mu_n^*(R) < \mu_n(R)$. By Lemma 4.7 we then have $R < 0.32$ and $h_n(0, \mu_n^*(R), R) > 0$. Hence $\tau = \tau(n, R) > 0$ in (4.12) and (4.13).

By (4.17)

$$\text{Re } P_n(R\sqrt{i\tau + 1}) > 0.$$

Substituting this estimate into (4.12), we obtain

$$\frac{n[n(n+1)/2 - 1]}{\mu_n^* R^5 P_0(R)} < P_1(R). \tag{4.38}$$

By the definition of μ_{n-1} (cf. (4.8))

$$\begin{aligned}
&\frac{n[n(n+1)/2 - 1]}{\mu_{n-1} R^5 P_0(R)} \\
&= \frac{n[n(n+1)/2 - 1]}{(n-1)[(n-1)n/2 + 1]} \{P_1(R) - P_{n-1}(R)\} \\
&= P_1(R) + \frac{1}{(n+1)(n-2)} \{(3n+2)P_1(R) - n(n+2)P_{n-1}(R)\} \\
&> P_1(R) + \frac{1}{(n+1)(n-2)} \Big\{(3n+2)P_1(0.4) - \frac{n(n+2)}{2n+1}\Big\} \\
&> P_1(R) \quad \text{for } 0 < R < 0.4;
\end{aligned}$$

here we use the fact $(3n+2)P_1(0.4) > 0.199(3n+2) = 0.597n + 0.398 > \{n(n+2)\}/(2n+1)$. Comparing the last inequality with (4.38) we conclude that $\mu_n^*(R) > \mu_{n-1}(R)$ for $n \geq 3$. □

From Theorem 4.8 and Lemma 4.9 we infer:

Theorem 4.10 *There holds:*

$$\mu_n^*(R) > \mu_2(R) \text{ for all } n \geq 3, \text{ and} \tag{4.39}$$

$$\mu_2^*(R) = \mu_2(R) \text{ if } R > R_2^*,\ \mu_2^*(R) < \mu_2(R) \text{ if } R < R_2^*. \tag{4.40}$$

We next prove

Lemma 4.11 *There exists a $\delta > 0$, depending only on μ and R, such that the real parts of the roots of $h_n(s, \mu, R)$ are less than $-\delta n^2$ for all sufficiently large n.*

Proof. The point of the theorem is that δ can be chosen independently of n. We choose δ so small that $\delta n^2 < (j_{n+1/2,1}/R)^2$, which is possible by (2.17). Consider a domain $D_n = \{s;\ -K_n < \text{Re } s < -\delta n^2,\ |\text{Im } s| < \widetilde{K}\}$, where $\widetilde{K}$ and K_n are arbitrarily large and choose K_n such that also (by (2.18)) $|K_n + 1 + (j_{n+1/2,m}/R)^2| > 1/R^2$ for all m. Let us decompose h_n as in (4.3), (4.2). Recalling (2.19) we have $|k_{1n}(s)| > |k_{2n}(s)|$ on ∂D_n if n is sufficiently large. As in the proof of Lemma 4.1, $h_n(s, \mu, R)$ has roots $\alpha_{nm} (m = 0, 1, 2, 3, \cdots)$ which lie in $D_\infty = \{s;\ \text{Re } s < -\delta n^2\}$ and satisfy

$$-1 - \Big(\frac{j_{n+1/2,m}}{R}\Big)^2 < \alpha_{nm} < -1 - \Big(\frac{j_{n+1/2,m-1}}{R}\Big)^2 \quad \text{if } m = 2, 3, \cdots \tag{4.41}$$

and the two additional roots α_{n1} and α_{n0}, real or complex, which lie in D_n. Since by Lemma 4.1 $h_n(s, \mu, R)$ has no additional roots, the assertion of Lemma 4.11 follows. □

We finally consider to the zeros of $h_1(s, \mu, R)$. Recall that

$$h_1(s, \mu, R) = \frac{s}{\mu R^2 P_0(R)} - P_1(R) + P_1(R\sqrt{s+1}), \tag{4.42}$$

and in particular $h_1(0, \mu, R) = 0$ for all μ, R.

Lemma 4.12 *Let*

$$\mu_1(R) = -\frac{2}{R^3 P_0(R) P_1'(R)}. \tag{4.43}$$

Then

(i) If $\mu < \mu_1(R)$ then, except for $s = 0$, all remaining zeros of $h_1(s, \mu, R)$ are real and negative.

(ii) If $\mu > \mu_1(R)$, then $h_1(s, \mu, R)$ has a positive real root.

Proof. From (4.42) we get

$$\frac{\partial}{\partial s} h_1(0, \mu, R) = \frac{1}{\mu R^2 P_0(R)} + \frac{R}{2} P_1'(R),$$

which is positive if $\mu < \mu_1(R)$. Similarly and $(\partial/\partial s)h_1(0, \mu, R) < 0$ if $\mu > \mu_1(R)$. Proceeding as in the proof of Theorem 4.8 we obtain the assertion of the lemma. □

Lemma 4.13

$$\mu_0^*(R) < \mu_1(R) \quad \text{for } 0 < R < \infty. \tag{4.44}$$

Proof. Let $h_0(i\tau, \mu, R) = 0$ with τ and $\mu = \mu_0^*(R)$ as in (4.12), (4.13) with $n = 0$. From (2.19) and (2.16) we deduce the inequality $-P_n'(r) > -P_{n+1}'(r)$. Hence

$$\frac{1}{\mu_1(R)R^2P_0(R)} = -\frac{1}{2}RP_1'(R) < -\frac{1}{2}RP_0'(R) = \sum_{m=1}^{\infty}\frac{2R^2}{(R^2+(m\pi)^2)^2}$$
$$< \sum_{m=1}^{\infty}\frac{2R^2}{R^4\tau^2+(R^2+(m\pi)^2)^2} = -\mathrm{Im}\frac{P_0(R\sqrt{i\tau+1})}{\tau} = \frac{1}{\mu_0^*(R)R^2P_0(R)},$$

by (4.17) and (4.13), and the lemma follows. □

5. Asymptotic stability and instability for the linearized problem

The following Laplace inverse transform formula is well known:

Lemma 5.1 *Suppose that $F(s)$ is holomorphic in Re $s \geq a$ and*

$$\lim_{K\to\infty}\sup\{|F(s)|;\ Re\ s \geq a, |s| \geq K\} = 0.$$

Suppose also that there exist an $\eta > 0$ and a constant c_0 such that

$$F(s) = c_0 s^{-1} + O(|s|^{-1-\eta}) \quad \text{for } s = a + i\tau, \quad \text{as } |\tau| \to \infty.$$

Then the integral

$$g(t) \equiv \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(s)e^{st}ds = \frac{e^{at}}{2\pi}\int_{-\infty}^{\infty} F(a+i\tau)e^{i\tau t}d\tau$$

defines the inverse Laplace transform of $F(s)$, i.e., $\widehat{g}(s) = F(s)$ for Re $s > a$, and furthermore,

$$|g(t)| \leq Ce^{at}.$$

Lemma 5.2 *Let $\mu < \mu_n^*(R)$ $(n \neq 1)$. Then there exists a small positive δ, depending only on μ, R, such that the following is true: If $Q_n(s)$ is a holomorphic function for Re $s \geq -\delta(n^2+1)$ such that $|Q_n(s)| \leq C/\sqrt{|s|+1}$, then*

$$\Big|\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{st}ds}{h_n(s,\mu,R)}\Big| + \Big|\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{Q_n(s)e^{st}ds}{h_n(s,\mu,R)}\Big| \leq Ce^{-\delta(n^2+1)t}, \tag{5.1}$$

where the constant C is independent of n.

Proof. We decompose h_n as follows: $h_n(s,\mu,R) = c_1(s + c(n) + k_n(s))$, where $c_1 k_n(s) = -P_1(R) + P_n(R\sqrt{s+1})$, c_1 is a positive constant, and $c(n) \approx c_2 n^3$ as $n \to \infty$, where c_2 is a positive constant. By (2.19), $|k_n(s)| \le$ const.$= c_3$ if Re $s > -\delta(n^2+1)$, provided δ is chosen so that $\delta(n^2+1)$ is smaller than $(j_{n+1/2,1}/R)^2$; by (2.17), such a choice of δ is possible.

By the Residue theorem,

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{st}ds}{s+c(n)} = e^{-c(n)t},$$

and then

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{e^{st}ds}{s+c(n)+k_n(s)} \\
&= \frac{1}{2\pi i}\Big[\int_{a-i\infty}^{a+i\infty} \frac{e^{st}ds}{s+c(n)} - \int_{a-i\infty}^{a+i\infty} \frac{k_n(s)e^{st}ds}{(s+c(n))(s+c(n)+k_n(s))}\Big] \\
&= e^{-c(n)t} - \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{k_n(s)e^{st}ds}{(s+c(n))(s+c(n)+k_n(s))} \;\equiv\; e^{-c(n)t} - J.
\end{aligned}$$

In the last integral we may change the contour of integration moving a to $-\delta(n^2+1)$, since $h_n(s,\mu,R)$ does not have a zero in the half plane Re $s \ge -\delta(n^2+1)$, by Lemma 4.11 (we may need to change δ, making it smaller) and the definition of $\mu_n^*(R)$. Hence

$$\begin{aligned}
|J| &\le \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-\delta(n^2+1)t}d\tau}{|(i\tau - \delta(n^2+1) + c(n))(i\tau - \delta(n^2+1) + c(n) + k_n(i\tau))|} \\
&\le Ce^{-\delta(n^2+1)t}\int_{-\infty}^{\infty} \frac{d\tau}{\tau^2+1} \le Ce^{-\delta(n^2+1)t};
\end{aligned}$$

and the first part of the (5.1) follows for n large, say $n \ge n_0$. Similarly, if $|Q_n(s)| \le C/\sqrt{|s|+1}$ for Re $s \ge -\delta(n^2+1)$, then

$$\begin{aligned}
\Big|\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{Q_n(s)e^{st}ds}{s+c(n)+k_n(s)}\Big| &= \Big|\frac{1}{2\pi i}\int_{-\delta(n^2+1)-i\infty}^{-\delta(n^2+1)+i\infty} \frac{Q_n(s)e^{st}ds}{s+c(n)+k_n(s)}\Big| \\
&\le Ce^{-\delta(n^2+1)t}\int_{-\infty}^{\infty} \frac{d\tau}{(\tau^2+1)^{3/4}},
\end{aligned}$$

and the second part of (5.1) follows for n large, say $n \ge n_0$. For each $n < n_0$ and $\ne 1$ the proof is similar provided $\mu < \mu_n^*(R)$. □

Theorem 5.3 *If $\mu < \mu_n^*(R)$, ($n = 0$ or $n \ge 2$) then for the solution $\rho(\theta,\varphi,t) = \rho_{n,m}(t)Y_{n,m}(\theta,\varphi)$ of (3.2)–(3.6) with initial condition*

$$\begin{aligned}
&\rho(\theta,\varphi,0) = \rho_{0,n,m}Y_{n,m}(\theta,\varphi) \quad (\rho_{0,n,m} \text{ constant}) \\
&w(r,\theta,\varphi,0) = w_{0,n,m}(r)Y_{n,m}(\theta,\varphi)
\end{aligned} \tag{5.2}$$

there holds:

$$|\rho_{n,m}(t)| \le C\Big(|\rho_{0,n,m}| + \frac{1}{(n+1)^{1/2}}\|w_{0,n,m}\|_{L^2}\Big)e^{-\delta(n^2+1)t} \quad \text{for } t>0, \tag{5.3}$$

where the constants C, δ depend on (R,μ) but are independent of (n,m).

Proof. Let

$$M_{n,m}(s) = \frac{1}{\mu R^2 P_0(R) h_n(s,\mu,R)}\Big\{\rho_{0,n,m} + \frac{\partial Q_{n,m}}{\partial r}(R)\Big\}, \tag{5.4}$$

where

$$Q_{n,m} = \frac{\mu}{s+1}w_{1n} - \psi_{1n} \quad \text{in } B_R \tag{5.5}$$

and w_{1n}, ψ_{1n} are determined by (3.19) and (3.22) with (ρ_0, w_0) replaced by $(\rho_{0,n,m}, w_{0,n,m})$. By Lemmas 3.2 and 3.3,

$$\Big|\frac{\partial Q_{n,m}}{\partial r}(R)\Big| \le \frac{C}{(n+1)^{1/2}}\frac{1}{|s+1|}\|w_{0,n,m}\|_{L^2}. \tag{5.6}$$

Using the equations for w_{1n} and ψ_{1n}, we find that, as $s+1 \to 0$, $(s+1)Q_{n,m}$ will satisfy a homogeneous elliptic system, and therefore

$$\lim_{s\to -1}(s+1)Q_{n,m}(r,s) \equiv 0,$$

so that $s=-1$ is not a pole for $Q_{n,m}$. Thus

$$\Big|\frac{\partial Q_{n,m}}{\partial r}(R)\Big| \le \frac{C}{(n+1)^{1/2}}\frac{1}{|s|+1}\|w_{0,n,m}\|_{L^2}. \tag{5.7}$$

Using Lemma 5.2 it follows that if $\mu < \mu_n^*$ then $M_{n,m}(s)$ is holomorphic in Re $s \ge -\delta$ for some $\delta > 0$, and δ is independent of n, by Lemma 4.11. Furthermore, using the form of $h_n(s)$, and the representation of $Q_{n,m}(r)$, and (5.6) it is easily seen that

$$M_{n,m}(s) = \frac{c_0}{s} + O\Big(\frac{1}{|s|^{3/2}}\Big) \quad \text{if Re } s \ge -\delta, \quad |s| \to \infty. \tag{5.8}$$

Hence, by Lemma 5.1 we deduce that the inverse Laplace transform

$$\rho_{n,m}(t) = \frac{1}{2\pi i}\int_{-\delta_1 - i\infty}^{-\delta_1 + i\infty} M_{n,m}(s)e^{st}ds \tag{5.9}$$

exists for any $0 < \delta_1 \le \delta$, and by Lemmas 5.1 and 5.2,

$$|\rho_{n,m}(t)| \le C\Big(|\rho_{0,n,m}| + \frac{C}{(n+1)^{1/2}}\|w_{0,n,m}\|_{L^2}\Big)e^{-\delta(n^2+1)t}.$$

We now introduce the solution (w,q) to (3.2)–(3.5) with $\rho(\theta,\varphi,t) = \rho_{n,m}(t)Y_{n,m}(\theta,\varphi)$. It remains to prove that $\rho(\theta,\varphi,t)$ satisfies the equation (3.6). The solution of (3.2)–(3.5) has the form

$$w(r,\theta,\varphi,t) = w_{n,m}(r,t)Y_{n,m}(\theta,\varphi),$$
$$q(r,\theta,\varphi,t) = q_{n,m}(r,t)Y_{n,m}(\theta,\varphi),$$

Note that $w_{n,m}$ satisfies the equation

$$\frac{\partial}{\partial t}w_{n,m} - \Delta w_{n,m} + \Big(\frac{n(n+1)}{r}+1\Big)w_{n,m} = 0.$$

Multiplying both sides by $w_{n,m}+\lambda\rho_{n,m}$ and integrating over B_R, we obtain

$$\|w_{n,m}(\cdot,t)\|_{L^2(B_R)} \le C(1+n^2)\Big(|\rho_{0,n,m}| + \frac{C}{(n+1)^{1/2}}\|w_{0,n,m}\|_{L^2}\Big)e^{-\delta(n^2+1)t},$$

and then, from the equation for $q_{n,m}$, we obtain

$$\|q_{n,m}(\cdot,t)\|_{H^2(B_R)} \le C(1+n^2)\Big(|\rho_{0,n,m}| + \frac{C}{(n+1)^{1/2}}\|w_{0,n,m}\|_{L^2}\Big)e^{-\delta(n^2+1)t}.$$

This allows us to take the Laplace transform in (3.2), (3.4) for Re $s > -\delta$ and, as in §2, we arrive at the formula (3.24) with $\widehat{\rho}_{n,m}$ replaced by $M_{n,m}(s)$.

On the other hand, the Laplace transform of $d\rho(\theta,\varphi,t)/dt - \mu(1-\widetilde{\sigma})\rho(\theta,\varphi,t)$ is equal to

$$\begin{aligned}
&[s\widehat{\rho}_{n,m}(s) - \rho_{0,n,m} - \mu(1-\widetilde{\sigma})\widehat{\rho}_{n,m}(s)]Y_{n,m}(\theta,\varphi)\\
&= \mu R^2P_0(R)\Big[M_{n,m}(s)\Big(\frac{s}{\mu R^2P_0(R)}\\
&\qquad -\frac{(1-\widetilde{\sigma})}{R^2P_0(R)}\Big) - \frac{1}{\mu R^2P_0(R)}\rho_{0,n,m}\Big]Y_{n,m}(\theta,\varphi)\\
&= \mu R^2P_0(R)\Big[M_{n,m}(s)\Big\{h_n(s) - P_n(R\sqrt{s+1})\\
&\qquad -\frac{n}{\mu R^5P_0(R)}\Big(\frac{n(n+1)}{2}-1\Big)\Big\} - \frac{1}{\mu R^2P_0(R)}\rho_{0,n,m}\Big]Y_{n,m}(\theta,\varphi)\\
&\qquad \text{(using (3.27): } \frac{(1-\widetilde{\sigma})}{R^2P_0(R)} = P_1(R)).
\end{aligned}$$

In view of (3.24) (with $\widehat{\rho}_{n,m}$ replaced by $M_n(s)$) and (3.25), the right-hand side is equal to

$$\begin{aligned}
&= \Big[\mu R^2P_0(R)M_{n,m}(s)h_n(s) - \frac{\partial\widehat{q}_{n,m}(r,s)}{\partial r}\Big|_{r=R} - \frac{dQ_{n,m}}{dr}\Big|_{r=R}\\
&\qquad -\rho_{0,n,m}\Big]Y_{n,m}(\theta,\varphi)\\
&= -\frac{\partial\widehat{q}_{n,m}(r,s)}{\partial r}\Big|_{r=R}Y_{n,m}(\theta,\varphi)\\
&= -\frac{\partial\widehat{q}(r,\theta,\varphi,s)}{\partial r}\Big|_{r=R}.
\end{aligned}$$

Thus the Laplace transform of $d\rho(r,\theta,\varphi,t)/dt - \mu(\overline{\sigma}-\widetilde{\sigma})\rho(r,\theta,\varphi,t)$ is equal to the Laplace transform of $-\partial q(R,\theta,\varphi,t)/\partial r$, and therefore (3.6) is satisfied. □

Consider next the case $n = 1$.

Theorem 5.4 *If $\mu < \mu_1(R)$, then for the solution $\rho(\theta,\varphi,t) = \rho_{1,m}(t)Y_{1,m}(\theta,\varphi)$ $(m=-1,0,1)$ of (3.2)–(3.6) with initial condition*

$$\begin{aligned} &\rho(\theta,\varphi,0) = \rho_{0,1,m}Y_{1,m}(\theta,\varphi) \quad (\rho_{0,1,m} \text{ constant}) \\ &w(r,\theta,\varphi,0) = w_{0,1,m}(r)Y_{1,m}(\theta,\varphi), \end{aligned} \tag{5.10}$$

there exists a constant a_m, determined uniquely by the initial data $\rho_{0,1,m}$ and $w_{0,1,m}(r)$, such that

$$|\rho_{1,m}(t) - a_m| \le C\Big(|\rho_{0,1,m}| + \|w_{0,1,m}\|_{L^2}\Big)e^{-\delta t} \quad \text{for } t > 0. \tag{5.11}$$

Proof. In this case $h_1(s,\mu,R)$ has one simple zero at $s = 0$, and all the rest of the zeros are real and negative.

Define $M_{1,m}$ be defined as in (5.4)–(5.5) with $n = 1$. As in the Theorem 5.3,

$$\rho_{1,m}(t) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} M_{1,m}(s)e^{st}ds \tag{5.12}$$

is well defined. The same argument shows that (3.2)–(3.5) define the corresponding $w_{1,m}$ and $q_{1,m}$ and (3.6) will be satisfied for $\rho_{1,m}(t)Y_{1,m}(\theta,\varphi)$.

It remains to establish the asymptotic estimate (5.11). Clearly, we can rewrite (5.12) as

$$\rho_{1,m}(t) = \frac{1}{2\pi i}\int_{-\delta_1-i\infty}^{-\delta_1+i\infty} M_{1,m}(s)e^{st}ds + \text{Res }\{M_{1,m}(s)e^{st};\ s=0\}. \tag{5.13}$$

Since $h_1(s,\mu,R)$ has a simple zero at $s = 0$,

$$\begin{aligned} a_m \equiv \text{Res }\{M_{1,m}(s)e^{st};\ s=0\} &= \frac{\Big[\Big(\rho_{0,n,m} + \frac{\partial Q_{n,m}}{\partial R}(R)\Big)e^{st}\Big]_{s=0}}{\mu R^2 P_0(R)(\partial h(0,\mu,R)/\partial s)} \\ &= \frac{\Big[\Big(\rho_{0,n,m} + \frac{\partial Q_{n,m}}{\partial R}(R)\Big)\Big]_{s=0}}{1+\mu R^3 P_0(R)P_1'(R)/2}, \end{aligned}$$

where the denominator is positive since $\mu < \mu_1(R)$. We can now proceed to complete the proof as in Theorem 5.3. □

Set

$$\mu_*(R) = \min(\mu_0^*(R), \mu_2^*(R)).$$

Theorem 5.5 *If $\mu < \mu_*(R)$, and $w_0 \in L^2(B_R)$, $\rho_0 \in L^2(\partial B_R)$. Then there exist constants a_m $(m = -1, 0, 1)$ and $\delta > 0$ such that*

$$\left|\rho(\theta,\varphi,t) - \sum_{m=-1}^{1} a_m Y_{1,m}\right| \le Ce^{-\delta t} \quad \text{for } 0 < t_0 < t < \infty. \tag{5.14}$$

Furthermore, the solutions belong to C^∞ for $t > 0$.

Proof. Write

$$\rho_0(\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m} \rho_{0,n,m} Y_{n,m}(\theta,\varphi),$$
$$w_0(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m} w_{0,n,m}(r) Y_{n,m}(\theta,\varphi),$$

then the assumptions imply

$$\sum_{n=0}^{\infty}\sum_{m} |\rho_{0,n,m}|^2 = \|\rho_0\|^2_{L^2} < \infty, \tag{5.15}$$
$$\sum_{n=0}^{\infty}\sum_{m} \|w_{0,n,m}\|^2_{L^2} = \|w_0\|^2_{L^2(B_R)} < \infty. \tag{5.16}$$

By Theorems 5.3 and 5.4, we can form a series solution

$$\rho(\theta,\varphi,t) = \sum_{n=0}^{\infty}\sum_{m} \rho_{n,m}(t) Y_{n,m}(\theta,\varphi)$$
$$w(r,\theta,\varphi,t) = \sum_{n=0}^{\infty}\sum_{m} w_{n,m}(r,t) Y_{n,m}(\theta,\varphi);$$

the estimates (5.3), (5.11), (5.15) and (5.16) imply that the last two series above are convergent in $L^2(B_R)$.

Clearly, for any positive integer k and any $t > t_0 > 0$,

$$\begin{aligned}
&\sum_{n=0,n\neq 1}^{\infty}\sum_{m}(1+n^{2k+1})|\rho_{n,m}(t)|^2 + \sum_{m=-1}^{1}|\rho_{1,m}(t) - a_m|^2 \\
&\le \sum_{n=0}^{\infty}\sum_{m}(1+n^{2k+1})[|\rho_{0,n,m}(t)|^2 + \|w_{0,n,m}\|^2_{L^2(B_R)}]e^{-2\delta(n^2+1)t} \\
&\le Ce^{-2\delta t},
\end{aligned}$$

where the constant C depends on t_0 and k. This means that $\rho(\cdot,\cdot,t) \in H^{k+1/2}(\partial B_R)$. Since k is arbitrary, we have, by embedding, $\rho \in C^\infty(\partial B_R)$ and (5.14) holds. We can similarly derive that w and q is C^∞ in (θ,φ). But then the elliptic and parabolic regularity imply that the solution is also C^∞ in r and in t for $t > 0$. □

Remark 5.1 The domain given by $r < R + \varepsilon \sum_{m=-1}^{1} a_m Y_{1,m}(\theta, \varphi)$ is not a sphere. But it is a sphere up to order $O(|\varepsilon|^2)$ after a translation of the origin of order $O(|\varepsilon|)$.

Having proved asymptotic stability in case $\mu < \min(\mu_0^*(R), \mu_2^*(R))$, we next prove asymptotic instability in case $\mu > \min(\mu_0^*(R), \mu_2^*(R))$.

Theorem 5.6 *For any $\mu > \mu_n^*$ $(n = 0, 2, 3 \cdots)$, (or $\mu > \mu_1(R)$ if $n = 1$) there exist initial conditions of the type*

$$\begin{aligned} &\rho(\theta, \varphi, 0) = \rho_{0,n,m} Y_{n,m}(\theta, \varphi) \quad (\rho_{0,n,m} \text{ constant}) \\ &w(r, \theta, \varphi, 0) = w_{0,n,m}(r) Y_{n,m}(\theta, \varphi) \end{aligned} \tag{5.17}$$

such that, for the corresponding solutions of (3.2)–(3.7), $\rho(\theta, \varphi, t) \to \infty$ exponentially fast for a sequence of points $t = t_n \to \infty$.

Proof. One can verify that

$$\begin{aligned} &\rho(t, \theta, \varphi) = e^{at} Y_{n,m}(\theta, \varphi), \\ &w(r, \theta, \varphi, t) = e^{at} u(r) Y_{n,m}(\theta, \varphi), \\ &q(r, \theta, \varphi, t) = e^{at} q(r) Y_{n,m}(\theta, \varphi), \end{aligned}$$

is a solution of (3.2)–(3.6) (with $\Delta u - [n(n+1)/r^2 + a + 1]u = 0$ in B_R, $u = -\lambda$ on ∂B_R, $\Delta q - [n(n+1)/r^2]q = -\mu u$ in B_R, $q = [n(n+1)/2 - 1]/R^2$ on ∂B_R) if and only if $h_n(a) = 0$. If $\mu > \mu_n^*$ ($\mu > \mu_1$ if $n = 1$) and the zero α_0 of $h_n(s)$ is real and positive, then, for the corresponding solution of (3.2)–(3.6), $\rho \to \infty$ exponentially fast. If α_1 and α_0 are complex numbers with $\alpha_1 = b + i\gamma$, then there are two linearly independent real-valued solutions with

$$\rho = (e^{bt} \cos \gamma t) Y_{n,m} \qquad \text{and } \rho = (e^{bt} \sin \gamma t) Y_{n,m},$$

and for each of them $\rho(t) \to \infty$ for a sequence $t_n \to \infty$. □

6. Comparison of μ_0^* with μ_2^*

The stability considered in this paper is different from the stability considered in [5]. In the case $n = 0$, we restrict ourselves to radially symmetric perturbations and this automatically implies that the limit spherical solution is *centered at origin*. On the other hand [5] deals with general perturbations, so that the center of the limit sphere may shift, as in (1.10). It is therefore be interesting to compare μ_0^* with μ_2^*.

Theorem 6.1 *There holds:*

$$\mu_0^*(R) < \mu_2^*(R) \quad \textit{for } R \textit{ small}, \tag{6.1}$$

$$\mu_0^*(R) > \mu_2^*(R) \quad \textit{for } R \textit{ large}. \tag{6.2}$$

From (4.36) and the asymptotic formula (2.9), we easily compute that

$$\mu_2^*(R) = \mu_2(R) = \frac{1}{2}R^{-2}[1 + O(R^{-1})] \quad \text{as } R \to \infty. \tag{6.3}$$

For $R < R_2^*$, $\mu_2^*(R)$ is given by solving (4.12) and (4.13). From (4.12), we have

$$\frac{4}{\mu_2^*(R)R^5P_0(R)} = P_1(R) - \text{Re } P_2(R\sqrt{i\tau+1}) < P_1(0) = \frac{1}{5}.$$

It follows that

$$\liminf_{R\to 0+} \mu_2^*(R)R^5 \geq 20/P_0(0) = 60. \tag{6.4}$$

Lemma 6.2 *There holds:*

$$\mu_0^*(R) < CR^{-4}, \tag{6.5}$$

for all R sufficiently small.

Proof. Recall that $\mu_0^*(R)$ is determined by solving the equations (4.12), (4.13)

$$P_1(R) = \text{ Re } P_0(R\sqrt{i\tau+1}\,), \quad \tau > 0, \tag{6.6}$$

$$\frac{\tau}{\mu_0^*(R)R^2P_0(R)} = -\text{ Im } P_0(R\sqrt{i\tau+1}\,), \tag{6.7}$$

Since

$$\begin{aligned}\text{Re } P_0(R\sqrt{i\tau+1}\,) - P_1(R) &> P_0(0) - P_1(R) - |P_0(R\sqrt{i\tau+1}\,) - P_0(0)| \\ &> \frac{1}{3} - \frac{1}{5} - C|R\sqrt{i\tau+1}|,\end{aligned}$$

where C is a constant independent of R and τ, it follows that if (R,τ) satisfies (6.6), then

$$R^2(\tau^2+1)^{1/2} > c_0, \tag{6.8}$$

for some positive c_0 independent of R and τ. This implies that $\tau \to \infty$ as $R \to 0$ and then, again, by (6.8),

$$R^2\tau > c_0, \tag{6.9}$$

with another positive constant c_0. Since the function $\coth(R\sqrt{i\tau+1})$ is uniformly bounded, and

$$\begin{aligned}\text{Re}P_0(R\sqrt{i\tau+1}\,) &= \Big|\text{Re}\Big(\frac{\coth(R\sqrt{i\tau+1})}{R\sqrt{i\tau+1}} - \frac{1}{R^2(i\tau+1)}\Big)\Big| \\ &\leq \frac{C_0}{R(\tau^2+1)^{1/4}}.\end{aligned} \tag{6.10}$$

Since $P_1(R)$ is monotone decreasing in R,

$$P_1(R) \geq P_1(2) > \frac{1}{6} \quad \text{if } R \leq 2.$$

From this inequality and (6.10) it follows that if

$$R < 2 \quad \text{and } R(\tau^2+1)^{1/4} \geq 6C_0,$$

then (6.6) cannot be satisfied.

Thus the solution to (6.6) must satisfy

$$R\tau^{1/2} < R(\tau^2+1)^{1/4} < 6C_0 \quad \text{if } R < 2. \tag{6.11}$$

Hence $R^2\tau$ is bounded from above and below by positive constants independent of R and τ. From the (2.19) we then get

$$-\text{ Im } P_0(R\sqrt{i\tau+1}) = 2\sum_{n=1}^{\infty} \frac{2R^2\tau}{R^4\tau^2 + (R^2+(n\pi)^2)^2} \geq c_*,$$

for some $c_* > 0$, so that, by (6.7)

$$R^2 P_0(R)\mu_0^*(R) \leq \tau/c_* < (36C_0^2)R^{-2}/c_*, \tag{6.12}$$

where the last inequality follows from (6.11). We thus obtain the inequality (6.5). □

Lemma 6.3

$$\liminf_{R\to\infty} \mu_0^*(R) > 0. \tag{6.13}$$

Proof. Since $\coth(R\sqrt{i\tau+1})$ is uniformly bounded. Hence

$$\begin{aligned} -\text{ Im } P_0(R\sqrt{i\tau+1}) &= -\text{ Im } \Big(\frac{\coth(R\sqrt{i\tau+1})}{R\sqrt{i\tau+1}} - \frac{1}{R^2(i\tau+1)}\Big) \\ &\leq \frac{C}{R(1+\tau^2)^{1/4}} + \frac{1}{R^2(1+\tau^2)^{1/2}} \\ &\leq \frac{C_*}{R} \quad \text{for } R \gg 1, \end{aligned}$$

where C^* is independent of R and τ. Using (6.7), we obtain

$$\mu_0^*(R) \geq \frac{R/C^*}{R^2 P_0(R)}, \tag{6.14}$$

and this implies (6.13). □

In the proof of Lemma 6.3 we have not used the relation (6.6).

Proof of Theorem 6.1. Compare the (6.4) with (6.5), and (6.3) with (6.13). □

Numerical calculations show that the two curves $\mu = \mu_2^*(R)$ and $\mu = \mu_0^*(R)$ intersect at a point $\overline{R} \sim 0.62207$. Hence

$$\mu^*(R) = \mu_0^*(R) < \mu_2^*(R) \begin{cases} < \mu_2(R) & \text{if } R < R_2^*, \\ = \mu_2(R) & \text{if } R_2^* < R < \overline{R} \end{cases}$$
$$\mu^*(R) = \mu_2^*(R) = \mu_2(R) \quad \text{if } R > \overline{R},$$

where $R_2^* \approx 0.15876$.

Remark 6.1 Consider a dynamical system

$$\frac{dx}{dt} = f(x, \mu)$$

and assume that $f(0, \mu) = 0$ for all values of the parameter μ. Assume also that $f(x, \mu) \neq 0$ if $x \neq 0$, $\mu < \mu_0$, but $f(x_0, \mu_0) = 0$ for some $x_0 \neq 0$. Assume also that all the eigenvalues of the matrix $f_x(x_0, \mu)$ for $\mu < \mu_0$ have negative real parts and either one or two of the eigenvalues cross the imaginary axis at $\mu = \mu_0$. In the first case we have a Liapounov-Schmidt bifurcation, and in the second case we have a Hopf bifurcation (see, for example, Weinberger [22]. In our situation, the Liapounov-Schmidt bifurcation occurs when $R > R^*$ (since the zero with maximal real part among the $h_n(s, \mu_*(R_S), R_S)$ crosses the imaginary axis at $s = 0$ with $n = 2$ and $\mu_*(R_S) = \mu_2(R_S)$). On the other hand if $R < R^*$, then two distinct conjugate zeros cross the imaginary axis at $\mu = \mu_*(R_S)$, and we expect this case to be like the typical pitch fork Hopf bifurcation, where each point on the two bifurcation branches determines a time-periodic solution which is asymtotically stable; furthermore the spherically symmetric stationary solution is not expected to remain asymptotically stable for $\mu > \mu_*(R_S)$. Hopf bifurcation does not manifest itself in the linearized problem which is dealt with in this paper. But we hope to consider it in our subsequent paper on the asymptotic stability and instability of the stationary problem for the system (1.1)–(1.5).

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